HW 8 Solution

1) $\psi(t) = e^{(t^2+3t)}$

$\psi'(t) = (2t+3)\, e^{(t^2+3t)}$

$\psi''(t) = 2e^{(t^2+3t)} + (2t+3)^2 e^{(t^2+3t)}$

mean $= E[X] = \psi'(0) = 3e^0 = 3$

$E[X^2] = \psi''(0) = 2e^0 + 3^2 e^0 = 2+9 = 11$

$Var[X] = E[X^2] - E[X]^2 = 11 - 3^2 = 2$

2) $\psi(t) = E[e^{tX}] = \sum_x e^{tx} P[X=x]$, so

$$f(x) = \begin{cases} 1/5 & x=1 \\ 2/5 & x=4 \\ 2/5 & x=8 \end{cases}$$

3) $Cov(aX+b, cY+d) = E\left[\left[(aX+b) - E(aX+b)\right]\left[(cY+d) - E(cY+d)\right]\right]$

$= E\left[(aX+b - aE[X] - b)(cY+d - cE[Y] - d)\right]$

$= E\left[a(X - E[X])(c(Y - E[Y]))\right]$

$= ac\, E\left[(X - E[X])(Y - E[Y])\right]$

$= ac\, Cov(X,Y)$

4a) $Var[X+Y+Z] = var(X) + Var(Y) + Var(Z) + 2\,Cov(X,Y) + 2\,Cov(Y,Z) + 2\,Cov(X,Z)$

$= 1+4+8+2(1+2-1)$

$= 13+4$

$= 17$

b) $Var[3X - Y - 2Z + 1] = Var[3X - Y - 2Z] = Var(3X) + Var(-Y) + Var(-2Z) + 2[Cov(3X,-Y) + Cov(-Y,-2Z) + Cov(3X,-2Z)]$

$= 9\,Var(X) + Var(Y) + 4Var(Z) - 6\,Cov(X,Y) + 4\,Cov(Y,Z) - 12\,Cov(X,Z)$

$= 9 + 4 + 32 - 6 + 8 + 12$

$= 59$

$$f_X(x) = \int_0^2 \frac{1}{3}(x+y)\,dy$$
$$= \frac{1}{3}\int_0^2 (x+y)\,dy$$
$$= \frac{1}{3} x \int_0^2 1\,dy + \frac{1}{3}\int_0^2 y\,dy$$
$$= \frac{1}{3}\cdot 2x + \frac{1}{3}\left[\frac{1}{2}y^2\right]_0^2$$
$$= \frac{2x}{3} + \frac{2}{3}$$
$$= \frac{2}{3}(x+1)$$

$$f_Y(y) = \int_0^1 \frac{1}{3}(x+y)\,dx$$
$$= \frac{y}{3}\int_0^1 1\,dx + \frac{1}{3}\int_0^1 x\,dx$$
$$= \frac{y}{3} + \frac{1}{3}\cdot\frac{1}{2}$$
$$= \frac{1}{3}\left(y+\frac{1}{2}\right)$$

$$E[X] = \int_0^1 x\, f_X(x)\,dx = \int_0^1 \left(\frac{2x^2}{3} + \frac{2x}{3}\right)dx$$
$$= \frac{2}{3}\int_0^1 (x^2+x)\,dx$$
$$= \frac{2}{3}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$
$$= \frac{2}{3}\left[\frac{1}{3}+\frac{1}{2}\right]$$
$$= \frac{2}{3}\cdot\frac{5}{6}$$
$$= \frac{10}{18}$$
$$\boxed{= \frac{5}{9}}$$

$$E[Y] = \int_0^2 \left(\frac{1}{3}y^2 + \frac{1}{6}y\right)dy$$
$$= \frac{1}{3}\int_0^2 \left(y^2 + \frac{1}{2}y\right)dy$$
$$= \frac{1}{3}\left[\frac{1}{3}y^3 + \frac{1}{4}y^2\right]_0^2$$
$$= \frac{1}{3}\left[\frac{8}{3}+1\right]$$
$$= \frac{1}{3}\cdot\frac{11}{3}$$
$$\boxed{= \frac{11}{9}}$$

$$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \frac{2}{3}\int_0^1 (x^3+x^2)\,dx$$
$$= \frac{2}{3}\left[\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_0^1$$
$$= \frac{2}{3}\left[\frac{1}{4}+\frac{1}{3}\right]$$
$$= \frac{2}{3}\cdot\frac{7}{12}$$
$$= \frac{7}{18}$$

$$E[Y^2] = \frac{1}{3}\int_0^2 \left(y^3 + \frac{1}{2}y^2\right)dy$$
$$= \frac{1}{3}\left[\frac{1}{4}y^4 + \frac{1}{6}y^3\right]_0^2$$
$$= \frac{1}{3}\left[4 + \frac{8}{6}\right]$$
$$= \frac{1}{3}\cdot\frac{32}{6}$$
$$= \frac{32}{18}$$
$$= \frac{16}{9}$$

5 con't)

$$Var[X] = E[X^2] - E[X]^2$$
$$= \frac{7}{18} - \left(\frac{5}{9}\right)^2$$
$$= \frac{7}{18} - \frac{25}{81}$$
$$= \frac{31.5-25}{81}$$
$$= \frac{6.5}{81}$$
$$= \frac{13}{162}$$

$$Var[Y] = E[Y^2] - E[Y]^2$$
$$= \frac{16}{9} - \left(\frac{11}{9}\right)^2$$
$$= \frac{16}{9} - \frac{121}{81}$$
$$= \frac{144-121}{81}$$
$$= \frac{23}{81}$$

$$E[XY] = \int_0^2 \int_0^1 \left(\frac{1}{3}xy(x+y)\right) dx\,dy = \frac{1}{3}\int_0^2\int_0^1 (x^2y + xy^2)\,dx\,dy$$
$$= \frac{1}{3}\int_0^2 \left[\frac{1}{3}x^3y + \frac{1}{2}x^2y^2\right]_0^1 dy$$
$$= \frac{1}{3}\int_0^2 \left[\frac{1}{3}y + \frac{1}{2}y^2\right] dy$$
$$= \frac{1}{3}\left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_0^2$$
$$= \frac{1}{3}\left[\frac{1}{6}(4+8)\right]$$
$$= \frac{2}{3}$$

$$cov(X,Y) = E[XY] - E[X]E[Y]$$
$$= 2/3 - \frac{5}{9}\cdot\frac{11}{9}$$
$$= 2/3 - \frac{55}{81}$$
$$= \frac{54-55}{81}$$
$$= \frac{-1}{81}$$

$$Var(2X - 3Y + 8) = Var[2X - 3Y]$$
$$= Var[2X] + Var[-3Y] + 2cov(2X, -3Y)$$
$$= 4Var[X] + 9Var[Y] - 12cov[X,Y]$$
$$= 4\left(\frac{13}{162}\right) + 9\left(\frac{23}{81}\right) - 12\left(-\frac{1}{81}\right)$$
$$= \frac{52 + 414 + 24}{162}$$
$$= \frac{490}{162}$$
$$= \frac{245}{81}$$