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# A review for Exam 1

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## Basic on Set Theory

Let  $A, B \subset S$ . Then,

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

$$A - B = \{x | x \in A \text{ and } x \notin B\}.$$

$$A \subset B \text{ means } x \in A \Rightarrow x \in B.$$

$$A = B \text{ if and only if } A \subset B \text{ and } B \subset A.$$

Let  $A \subset S$ . Then,

$$(A^c)^c = A.$$

$$\emptyset^c = S.$$

$$S^c = \emptyset.$$

$$A \cup A^c = S.$$

$$A \cap A^c = \emptyset.$$

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Let  $A, B, C \subset S$ . Then,

$$A \cup B = B \cup A.$$

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

$$A \cap B = B \cap A.$$

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

$$(A^c \cap B^c) = A^c \cup B^c$$

$$(A^c \cup B^c) = A^c \cap B^c$$

## Definition of Probability

**Definition** Suppose  $A_1, A_2, \dots \subset S$  are infinite sequence of events. Then we say  $A_1, A_2, \dots$  are disjoint iff

$$A_i A_j = \emptyset, \forall i, j \text{ with } i \neq j.$$

**Definition** A probability  $P$  is a function from the set of all possible events in  $S$  to  $\mathbb{R}$  such that

$$P(A) \geq 0 \quad \forall A \subset S,$$

$$\text{if } A_1, A_2, \dots \subset S \text{ are disjoint, } P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i),$$

$$P(S) = 1.$$

## Some Theorems

**Thm**

$$P(\emptyset) = 0.$$

**Thm** Suppose  $A_1, A_2, \dots, A_n \subset S$  are finite sequence of disjoint events.

Then,

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i).$$

**Thm**  $\forall A \subset S,$

$$P(A^c) = 1 - P(A) \text{ and } 0 \leq P(A) \leq 1.$$

**Thm**  $\forall A, B \subset S$  such that  $A \subset B,$

$$P(A) \leq P(B).$$

## Combinatorial Methods

**Definition** A permutation of order  $n$ ,  $S_n$ , is an arrangement or ordering of  $n$  objects.

**Definition** An  $r$  permutation of order  $n$ ,  $S_n^r$ , is an arrangement using  $r$  out of  $n$  objects.

**Definition** An  $r$  combination of  $n$  distinct objects is an unordered selection or subset of  $r$  out of  $n$  objects.

We write

$$P_{n,r} = \# \text{ of } S_n^r = \frac{n!}{(n-r)!},$$

$$C_{n,r} = \# \text{ of } r \text{ combinations of } n \text{ distinct objects} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

## Binomial Coefficients

**Definition**  $C_{n,r}$  are called binomial coefficients.

**Thm** (Binomial Theorem)  $C_{n,i}$  are coefficients of  $x^i$  in the polynomial  $(1+x)^n$ . In other words,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

**Note**  $C_{n,0} = C_{n,n} = 1$ .



## Binomial Identities

$$\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m},$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

$$\sum_{i=0}^n \binom{n}{i} = 2^n,$$

$$\sum_{i=0}^r \binom{n+i}{i} = \binom{n+r+1}{r},$$

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n},$$

## Binomial Identities cont....

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r},$$

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r},$$

$$\sum_{k=s-n}^{m-r} \binom{m-k}{r} \binom{n+k}{s} = \binom{m+n+1}{r+s+1}.$$

## Multinomial Coefficients

**Definition** A multinomial coefficient is defined by  $\frac{n!}{n_1!n_2!\cdots n_k!}$  where  $n_1 + n_2 + \cdots + n_k = n$  and  $n_i \geq 0$  integer for all  $i = 1, 2, \cdots, k$ . It is denoted by

$$\binom{n}{n_1, n_2, \cdots, n_k}.$$

### Thm (Multinomial Theorem)

For all numbers  $x_1, x_2, \cdots, x_k$  and each positive integer  $n$ , we have

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{n_1, n_2, \cdots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where the summand extends over all possible combinations of nonnegative integers  $n_1, n_2, \cdots, n_k$  such that  $n_1 + n_2 + \cdots + n_k = n$ .

## Probability of a union of events

**Thm** Suppose  $A_1, A_2, \dots, A_n \subset S$  are finite sequence of events. Then

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i < j} P(A_i A_j) + \sum_{i < j < k} P(A_i A_j A_k) \\ - \sum_{i < j < k < l} P(A_i A_j A_k A_l) + \dots (-1)^{n+1} P(A_1 A_2 \dots A_n).$$

## Conditional Probability

**Definition** Suppose  $A, B \subset S$ . The conditional probability of  $A$  given  $B$ ,  $P(A|B)$ , is a probability that  $A$  occurs after  $B$  occurs.

**Note** If  $A, B \subset S$  such that  $P(B) > 0$ , then

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

**Note** (Multiplication Rule)

$$P(AB) = P(B)P(A|B).$$

## Independent Events

**Definition**  $A, B \subset S$  are independent iff  $P(A)P(B) = P(AB)$ .

**Thm** If  $A, B \subset S$  are independent, then  $A, B^c$  are independent.

**Definition**  $A_1, A_2, \dots, A_n \subset S$  are independent iff for every subsets  $A_{i_1}, A_{i_2}, \dots, A_{i_j}$  of  $j$  of these events,

$$P(A_{i_1}A_{i_2} \cdots A_{i_j}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_j}).$$

**Definition**  $A_1, A_2, \dots, A_n \subset S$  are pairwise independent iff for every  $i, j$  with  $i \neq j$

$$P(A_iA_j) = P(A_i)P(A_j).$$

## Independent Events and Conditional Prob

**Note**  $A, B \subset S$  are independent iff  $P(A|B) = P(A)$ .

**Definition** Let  $A_1, A_2, \dots, A_n, B \subset S$ . We say Let  $A_1, A_2, \dots, A_n$  are conditionally independent given  $B$  iff for every subset  $A_{i_1}, A_{i_2}, \dots, A_{i_j}$  of  $j$  of these events,

$$P(A_{i_1}A_{i_2} \cdots A_{i_j}|B) = P(A_{i_1}|B)P(A_{i_2}|B) \cdots P(A_{i_j}|B).$$

**Thm** Suppose that  $A_1, A_2, B \subset S$  such that  $P(A_1B) > 0$ . Then  $A_1, A_2$  are conditionally independent given  $B$  iff  $P(A_2|A_1B) = P(A_2|B)$ .

## Law of Total Probability

**Definition** A collection  $\{B_i\}_{i=1}^{\infty}$  of disjoint events for which  $\cup_{i=1}^{\infty} B_i = S$  is called a partition of the sample space  $S$ .

**Thm** (Law of Total Probability)

For any partition of  $S$ ,  $\{B_i\}_{i=1}^{\infty}$ , for any event  $A \subset S$ , we have

$$P(A) = \sum_{i=1}^{\infty} P(AB_i) = \sum_{i=1}^{\infty} P(A|B_i)P(B_i).$$



## Law of Total Probability

**Thm** (Conditional version of Law of Total Probability)

For any partition of  $S$ ,  $\{B_i\}_{i=1}^{\infty}$ , for any event  $A, C \subset S$ , we have

$$P(A|C) = \sum_{i=1}^{\infty} P(AB_i|C) = \sum_{i=1}^{\infty} P(A|B_iC)P(B_i|C).$$

## Bayes' Theorem

### Thm (Bayes' Theorem)

Suppose  $\{B_i\}_{i=1}^{\infty}$  is a partition of  $S$  and  $A \subset S$  for which  $P(A) > 0$ . Then, for any event  $B_i$  with  $P(B_i) > 0$ , we have:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(B_j)P(A|B_j)}.$$

**Definition**  $P(B_i)$  in the equation above is called a prior probability and  $P(B_i|A)$  in the equation above is called a posterior probability.