

STA 321

Spring 2014

Lecture 19

Thursday, April 8th

- **Significance Tests for Mean and Proportion**
- **Power Calculations**

Type I and Type II Errors

- Terminology:
 - **Alpha** = Probability of a Type I error
 - **Beta** = Probability of a Type II error
 - **Power** = $1 - \text{Probability of a Type II error}$
- The smaller the probability of Type I error, the larger the probability of Type II error and the smaller the power
- If you ask for very strong evidence to reject the null hypothesis, it is more likely that you fail to detect a real difference

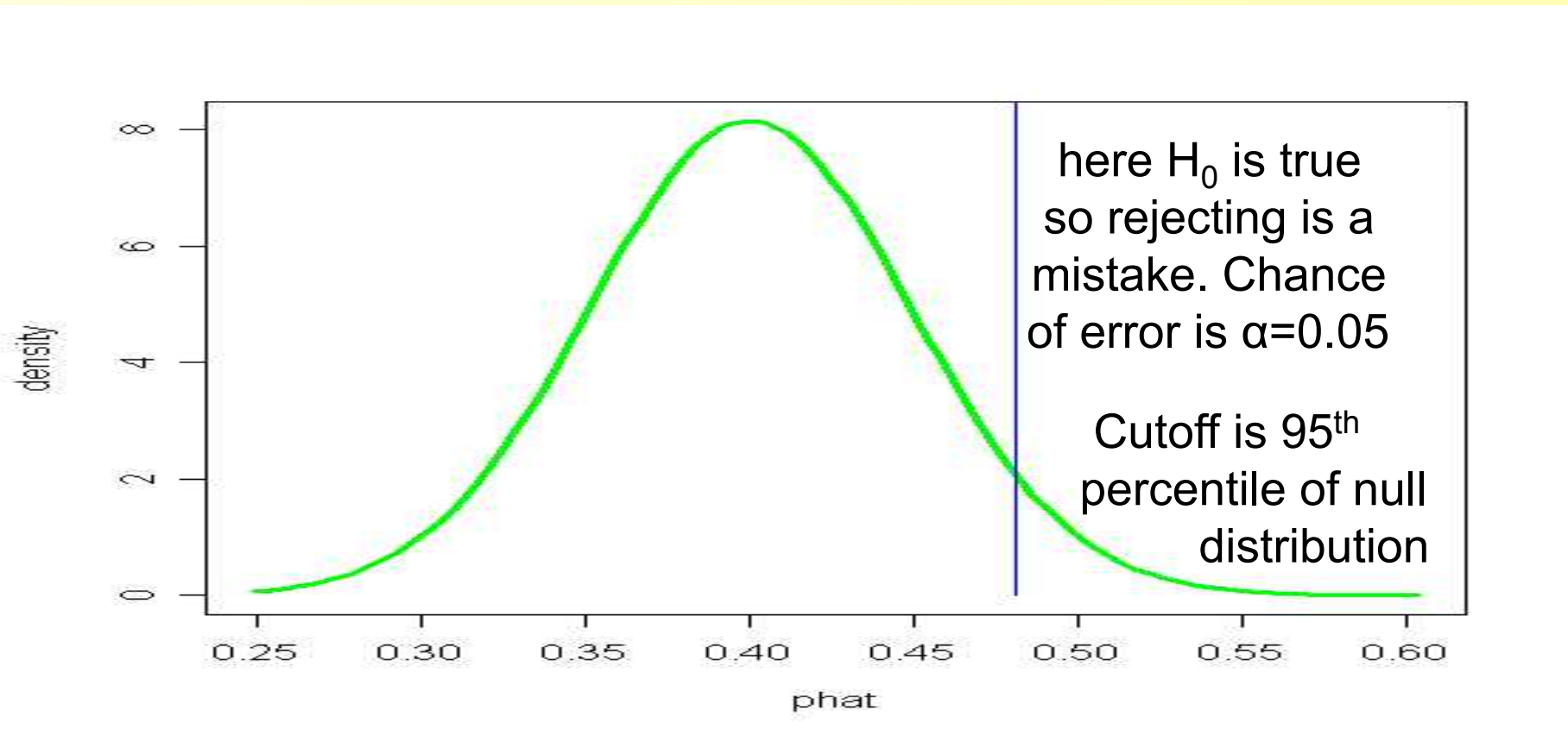
Power – Why Important?

- Example
 - A pharmaceutical company knows that the old treatment for a disease cures 40% of the people.
 - They hope their new treatment is better.
 - They hope they can get the cure rate to 45%.

Power Example Continued

- Our hypothesis test will test the null hypothesis $H_0 : p=0.4$ (the old treatment proportion) against $H_1 : p>0.4$ (we want our treatment to do better, hence this alternative).
- We intend to give the treatment to 100 people, and using $\alpha=0.05$
- Cutoff?
- 95th percentile of the null distribution.
- $Z=1.64$, thus $Y = (0.049)(1.64) + 0.4 = 0.4804$

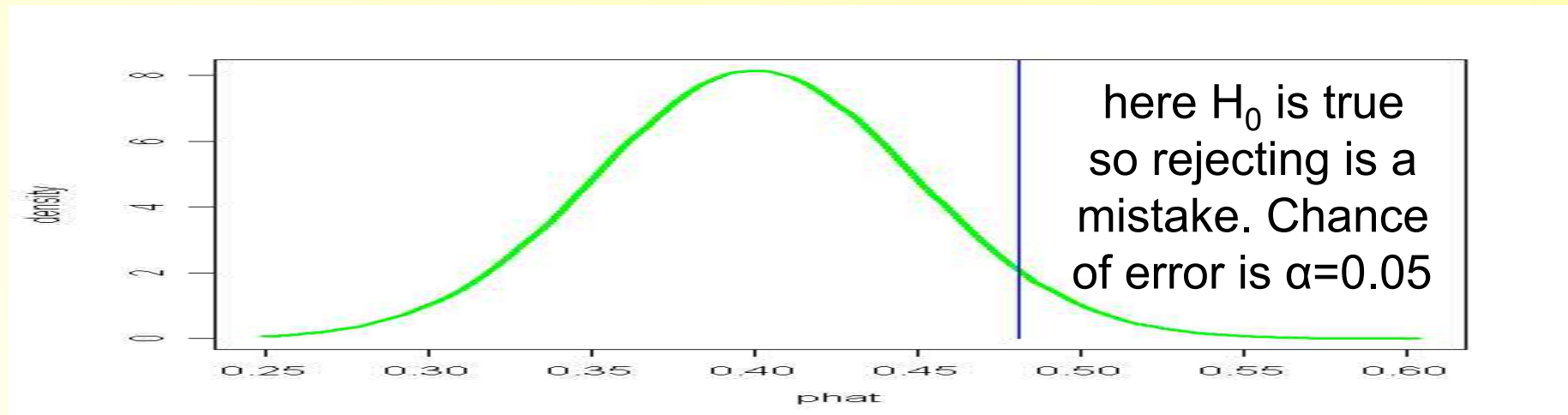
Null Distribution Centered at $p_0=0.4$



Can this Experiment Find Anything?

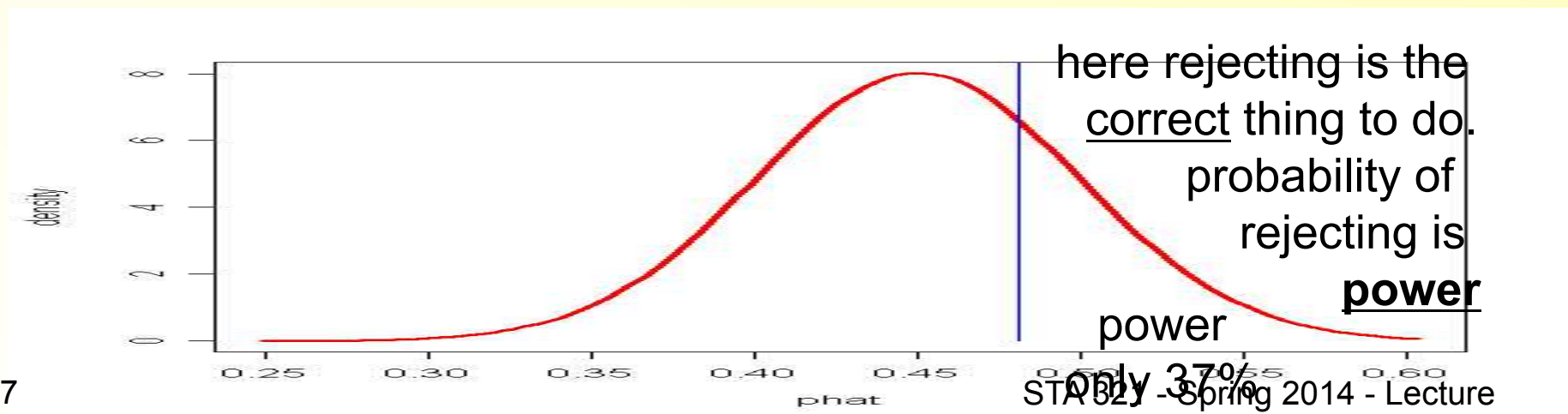
- We only are **guessing** our treatment can get the cure rate to 45%.
- What is the power for 45%?
- Remember, the power is the chance that, **when $p=0.45$** , we reject H_0 (the right decision in that case).
- We reject when $\hat{p} > 0.4804$.
- Thus, we need the probability that \hat{p} is greater than 0.4804, **given** $p=0.45$.

Green curve is distribution for $p=0.4$
Red curve is distribution for $p=0.45$



do not reject this side of blue line

reject this side of blue line



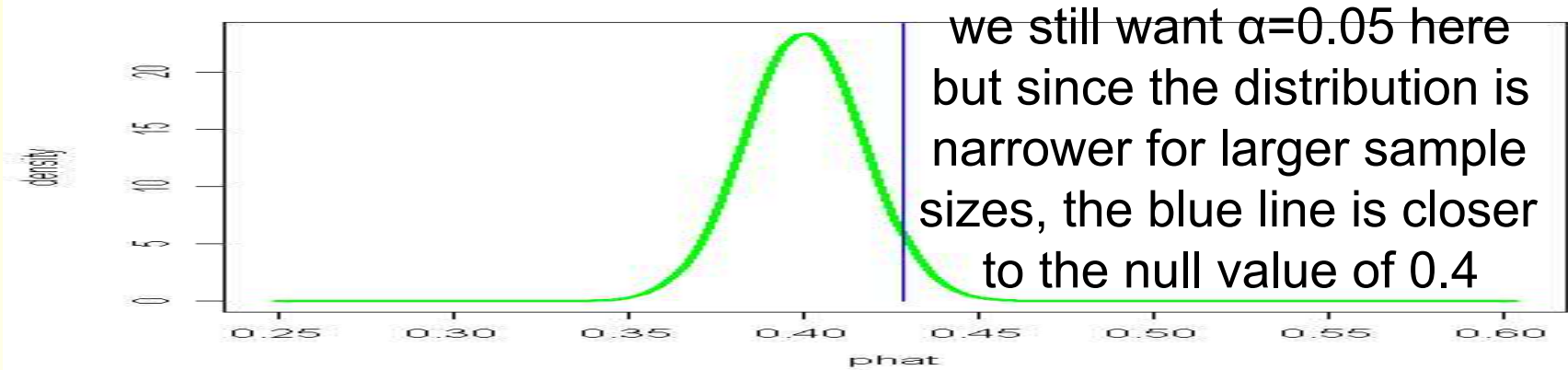
What This Means...

- Suppose our treatment works (this is the assumption under which the power is calculated).
- Then we only have a 37.09% chance of getting a “reject H_0 ” conclusion.
- That is not great. We could have a beneficial treatment and miss it.
- Solution – choose a higher sample size.

How Does a Larger Sample Size Help?

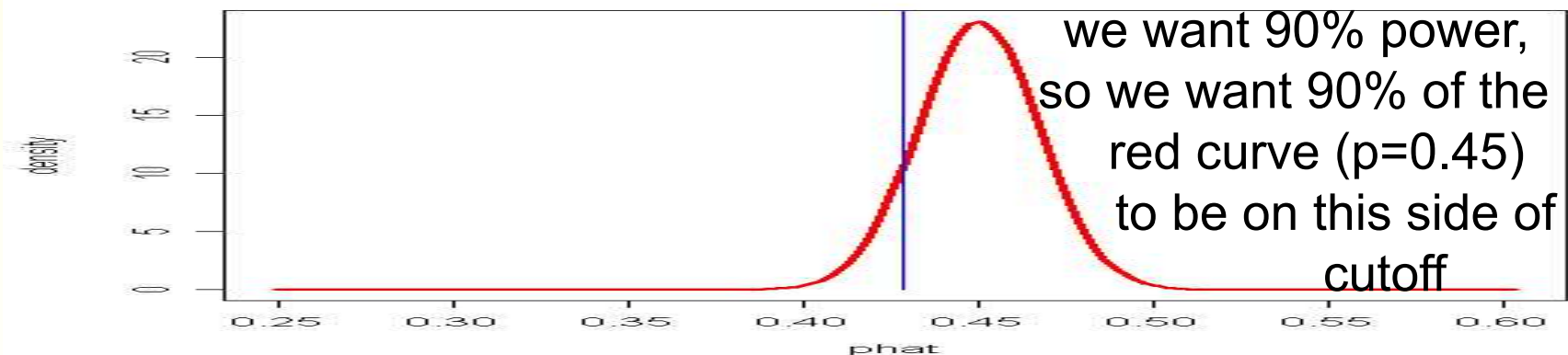
- A larger sample size reduces the standard deviations of both distributions, making them overlap less.
- Because the null distribution is narrower, the cutoff gets closer to 0.4, the null value.
- In fact, given a particular power, we can find an appropriate sample size.
- Suppose we want 90% power for $p=0.45$

Our goal – to get 90% power for $p=0.45$

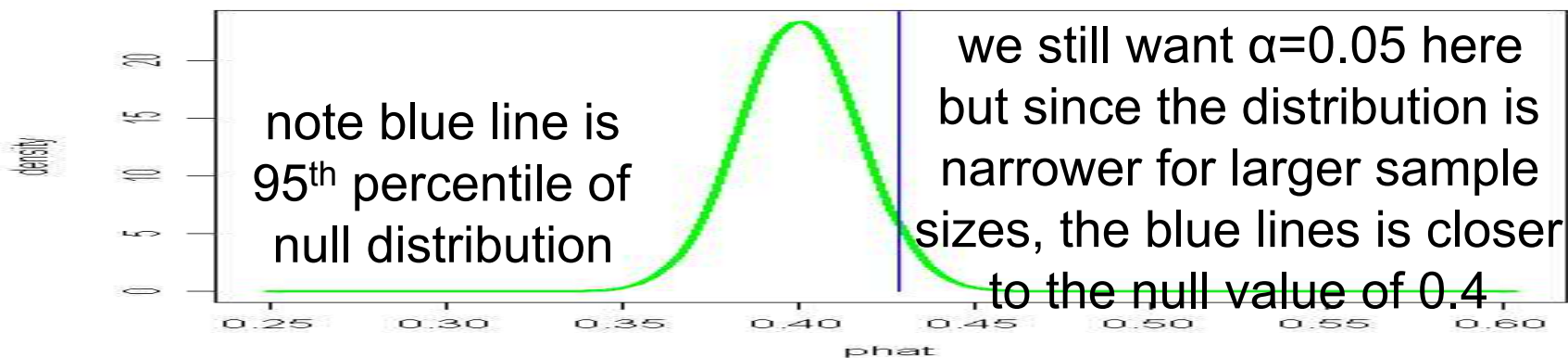


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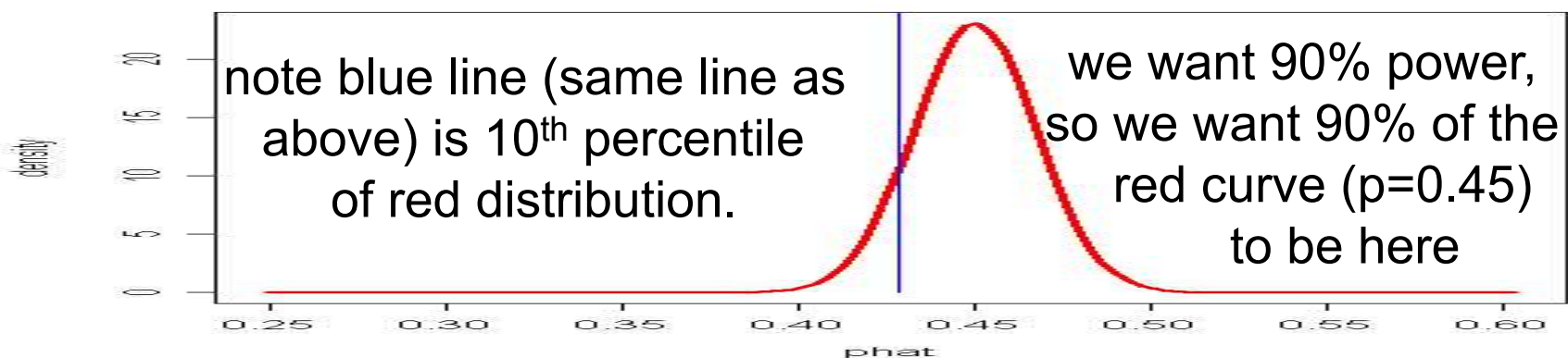


Our Goal – Get 90% Power for $p=0.45$



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reject this side of blue line



What Sample Size Should We Choose?

- The previous graph shows:
 - We need to equate the 95th percentile (**Z=1.64**) of the null distribution (using $p=0.40$) to the 10th percentile (**Z=-1.28**) of the distribution using $p=0.45$
 - With $p=0.40$, the standard deviation of \hat{p} is $\sqrt{0.40 \cdot 0.60/n} = 0.4899/\sqrt{n}$
 $\hat{p} \sim N(0.40, 0.4899/\sqrt{n})$
 - With $p=0.45$, the standard deviation of \hat{p} is $\sqrt{0.45 \cdot 0.55/n} = 0.4975/\sqrt{n}$
 $\hat{p} \sim N(0.45, 0.4975/\sqrt{n})$

The dreaded power equation

- With $p=0.4$, $\hat{p} \sim N(0.40, 0.4899/\sqrt{n})$
- With $p=0.45$, $\hat{p} \sim N(0.45, 0.4975/\sqrt{n})$
- We are equating the 95th percentile of a distribution ($Z=1.64$) to the 10th percentile of another distribution ($Z=-1.28$)
- The 95th percentile when $p=0.40$ is
 $(0.4899/\sqrt{n}) \cdot (1.64) + 0.40$
- The 10th percentile when $p=0.45$ is
 $(0.4975/\sqrt{n}) \cdot (-1.28) + 0.45$

Power Equation

- We need to solve for the n that satisfies

$$\frac{0.4899(1.645)}{\sqrt{n}} + 0.4 = \frac{0.4975(-1.28)}{\sqrt{n}} + 0.45$$

$$\frac{1.4427}{\sqrt{n}} = 0.05$$

- $n=832.5$, so n must be at least 833

Cheating....

- If you want to cheat, you could think “can’t I just test 100 people, and if I don’t find anything go to 200, and if I don’t find anything go to 300, etc.”
- This is called “testing to a foregone conclusion”, and is not good science.
- You run a much bigger chance of a type I error than you claim to be.
- The way to do this right is called “sequential analysis” (not studied in this course).

Another example with a 2-sided alternative

- One of the most basic tests of a random number generator is that it produces numbers in the right frequency (more complicated tests involve verifying no discernible pattern is present in the numbers)
- One random number generator is supposed to produce successes and failures in the ratio of 20% successes and 80% failures.
- Thus, $H_0 : p=0.2$ and $H_1 : p \neq 0.2$, since being off in either direction is bad.

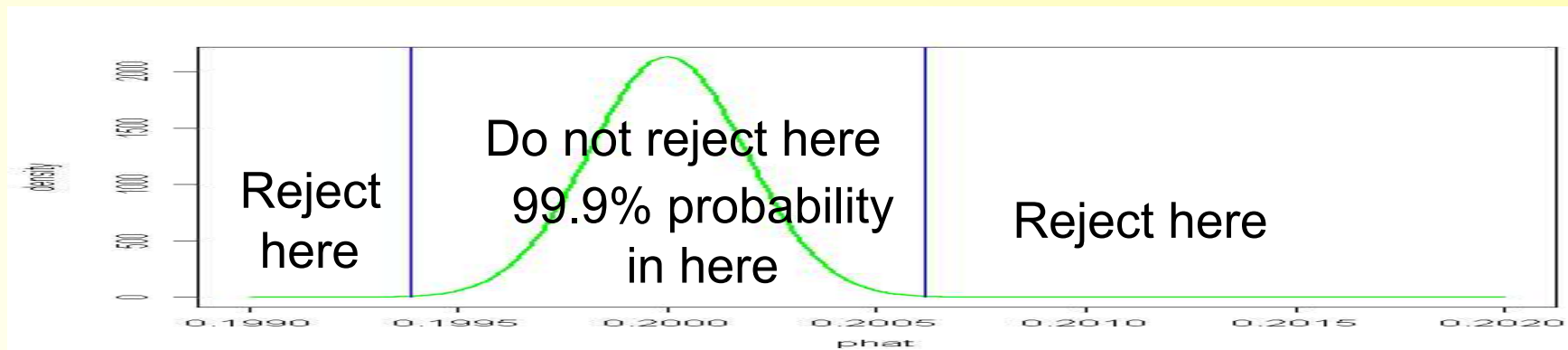
Example setup, continued

- We want to be quite sure of our random number generator (plus, generating large samples is cheap), so we want $\alpha=0.001$ and 98% power for $p=0.201$. Thus, we want to be quite likely to determine even a small difference from $p=0.2$.
- The resulting sample size will be quite large.

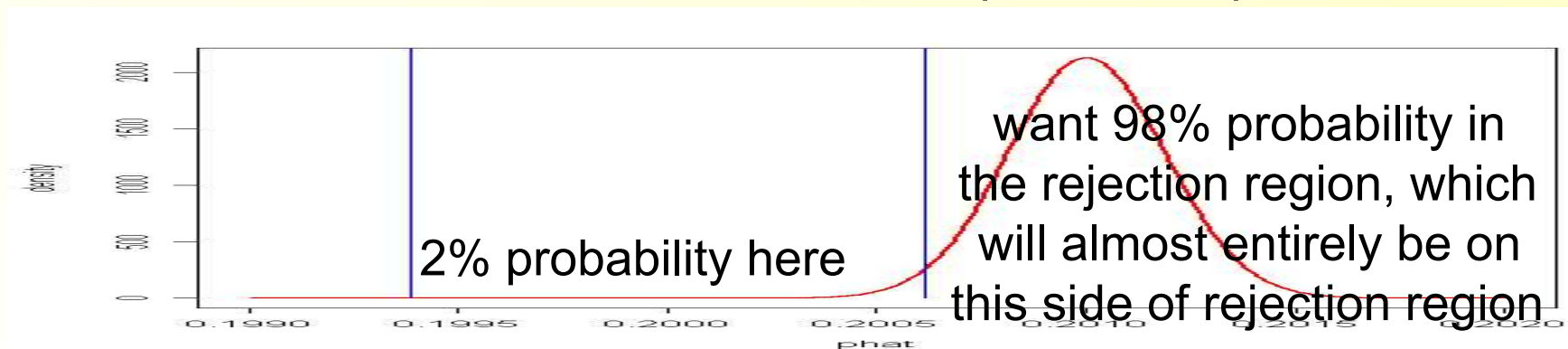
Null and Alternative distributions

- The null distribution ($p=0.2$) is normal with mean 0.2 and standard deviation $\sqrt{0.2*0.8/n} = 0.4/\sqrt{n}$
- The alternative distribution ($p=0.201$) is normal with mean 0.201 and standard deviation $\sqrt{0.201*0.799/n} = 0.400748/\sqrt{n}$
- We have a 2-sided alternative, so we reject in both directions from $p_0=0.2$

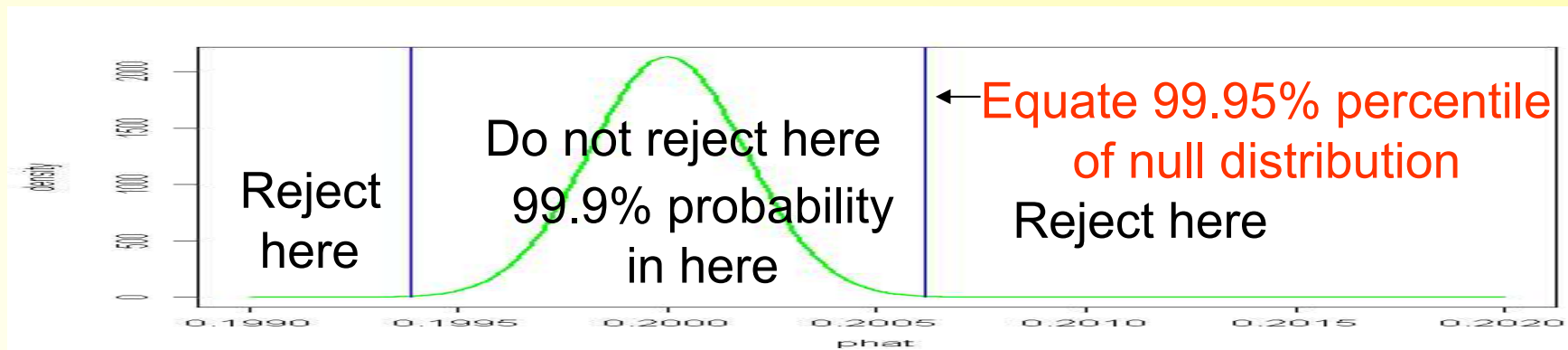
We want $\alpha=0.001$ and 98% power at $p=0.201$



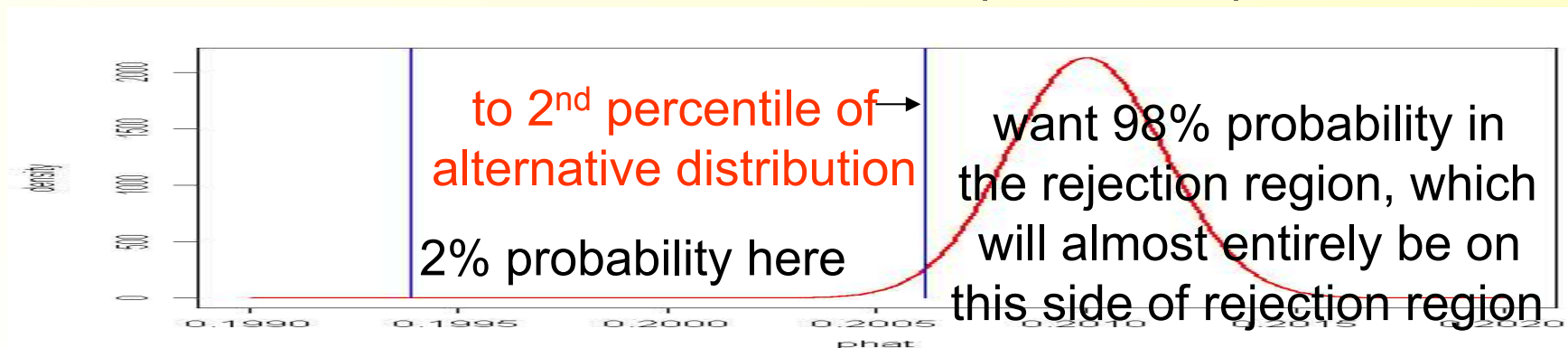
Only the right cutoff really matters for the power calculation (this is the cutoff in the direction from $p=0.200$ to $p=0.201$)



We want $\alpha=0.001$ and 98% power at $p=0.201$



Only the right cutoff really matters for the power calculation (this is the cutoff in the direction from $p=0.200$ to $p=0.201$)



Power equation

- We need to equate the 99.95% percentile of the null distribution to the 2nd percentile of the alternative distribution.
- The 99.95% percentile corresponds to $Z=3.29$ (nearby values acceptable). For the null distribution, this Z corresponds to $(3.29)(0.4/\sqrt{n}) + 0.20 =$
 $0.200 + 1.316/\sqrt{n}$
- The 2nd percentile corresponds to $Z=(-2.05)$. For the alternative distribution, this Z corresponds to $(-2.05)(0.400748)/\sqrt{n} + 0.201 =$
 $0.201 - 0.82153/\sqrt{n}$

Solving for n

- We equate
 $0.200 + 1.316/\sqrt{n} = 0.201 - 0.82153/\sqrt{n}$
- $2.13753/\sqrt{n} = 0.001$
- $2137.53 = \sqrt{n}$
- $n = 4,569,035$
- This is huge, but remember we want virtually certainty of detecting a very small difference.

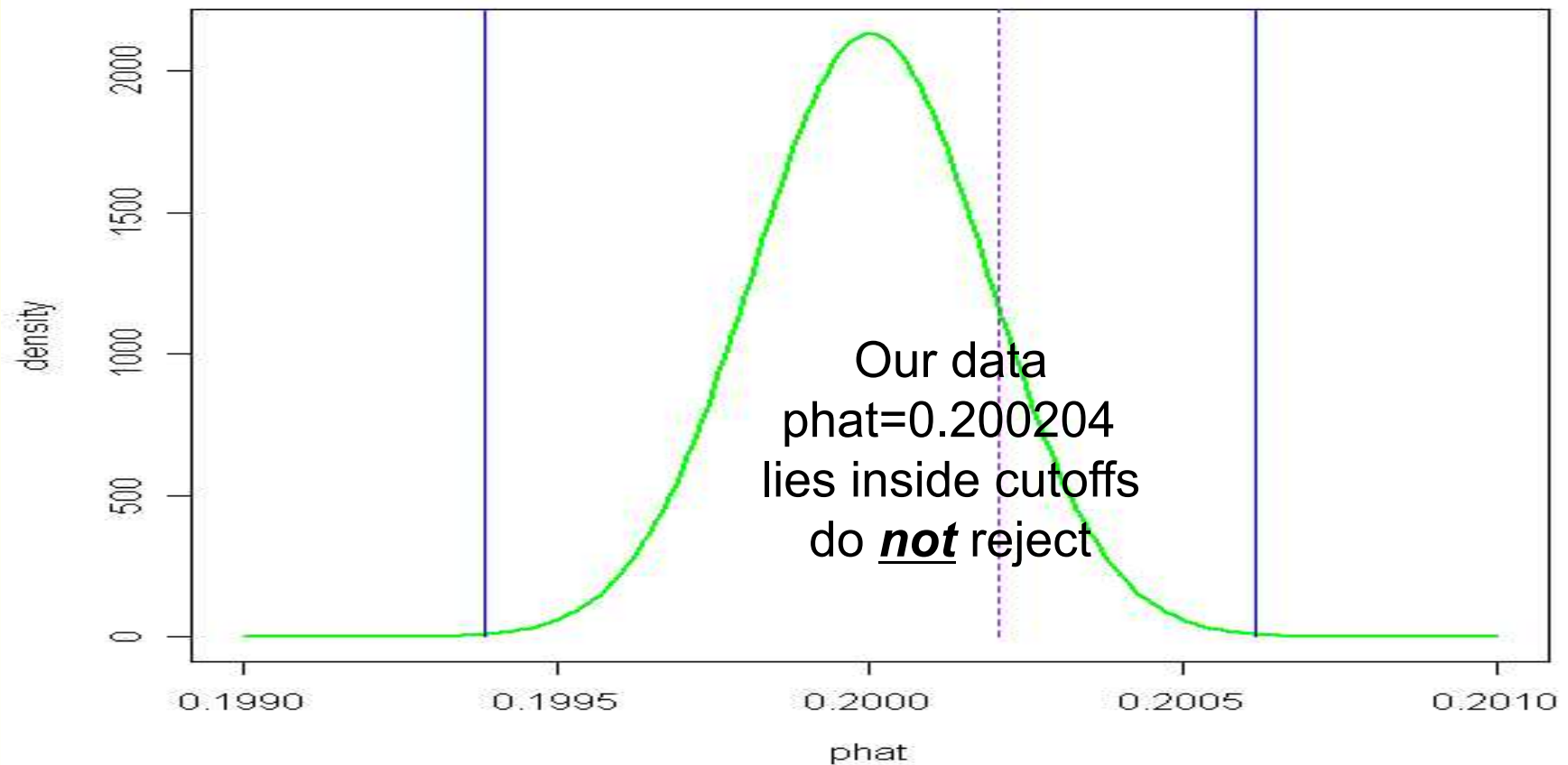
My experimental data

- As I said, computer random number are cheap, so we generated 5,000,000 of them, where we are supposed to get $p=0.2$
- I “rolled” 1,001,020 successes, which resulted in $\hat{p}=1001020/5000000 = 0.200204$
- Let's conduct the hypothesis test of $H_0 : p=0.2$ and $H_1 : p \neq 0.2$ and determine the result

The null distribution

- The null distribution is normal with mean 0.2 and standard deviation $\sqrt{(0.2 \cdot 0.8 / 5000000)} = 0.0001789$
- We are using $\alpha = 0.001$ and conducting a 2-sided test, so we are looking for the 0.05% and 99.95% percentiles, corresponding to $Z = (\pm 3.29)$.
- These percentiles are
 $(-3.29)(0.0001789) + 0.2 = 0.19994$ and
 $(3.29)(0.0001789) + 0.2 = 0.20006$

Reject outside (0.1994, 0.2006)



Conclusion

- So it appears our random number generator in R looks ok.
- As mentioned earlier, this is the EASIEST test for a random number generator to pass.
- Other tests involve trying to avoid patterns, which are not supposed to be present in random data.

Review of possible situations

- We always have a hypothesis of the form $H_0 : p=p_0$. In a sample size calculation, we are given an α , the alternative hypothesis, and a desired power POW for a specified p_1 in the alternative distribution.
- For example, “we are testing $H_0 : p=0.6$ against $H_1 : p>0.6$. We want to use $\alpha=0.05$ and achieve 90% power at $p=0.7$.”

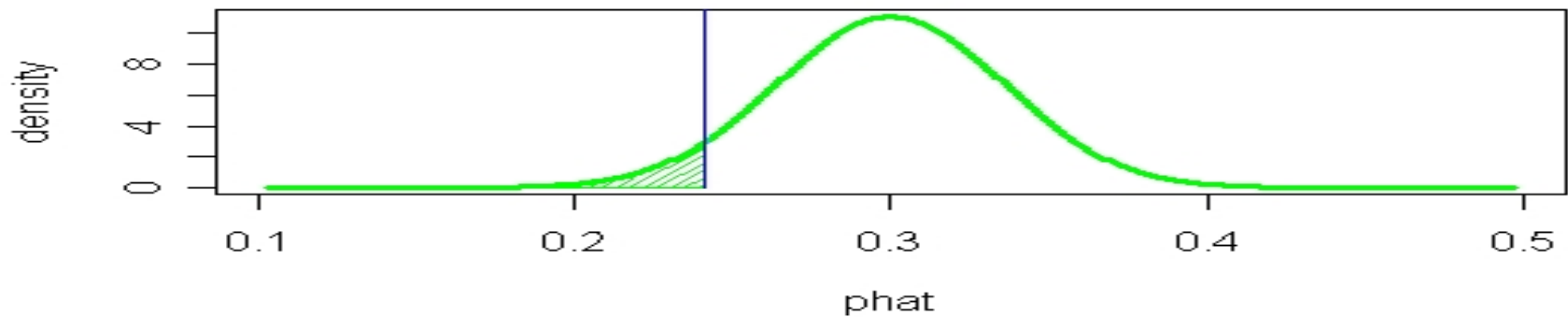
Review of possible situations

- There are three possible alternative hypotheses, $H_1 : p < p_0$, $H_1 : p > p_0$, or $H_1 : p \neq p_0$.
- For $H_1 : p \neq p_0$, the alternative value p_1 may be either less than p_0 or greater than p_0 .
- Thus, we have four possible scenarios.
- In each scenario, power is computed by equating a particular percentile of the null distribution to a percentile of the alternative distribution.

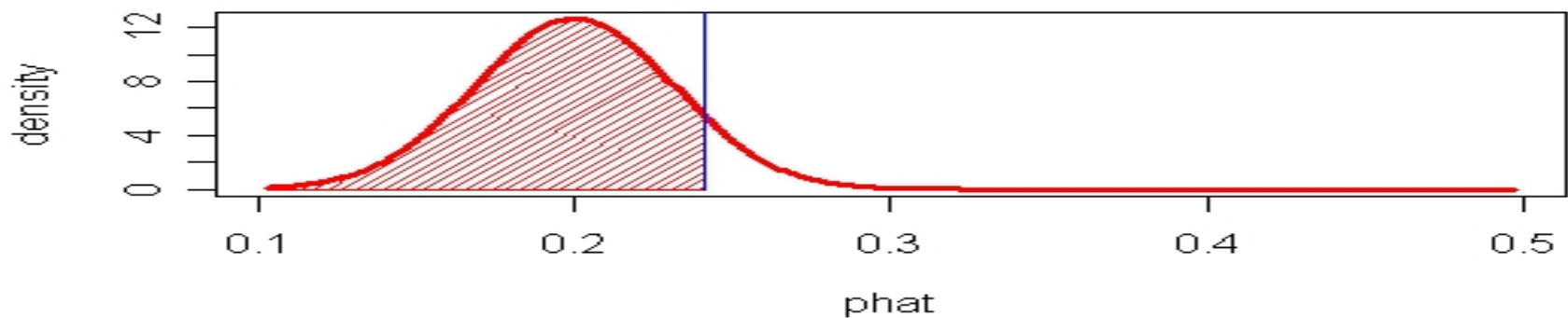
Review of possible situations

- The null distribution is always normal with mean p_0 and standard deviation $\sqrt{p_0(1-p_0)/n}$.
- The alternative distribution is always normal with mean p_1 and standard deviation $\sqrt{p_1(1-p_1)/n}$

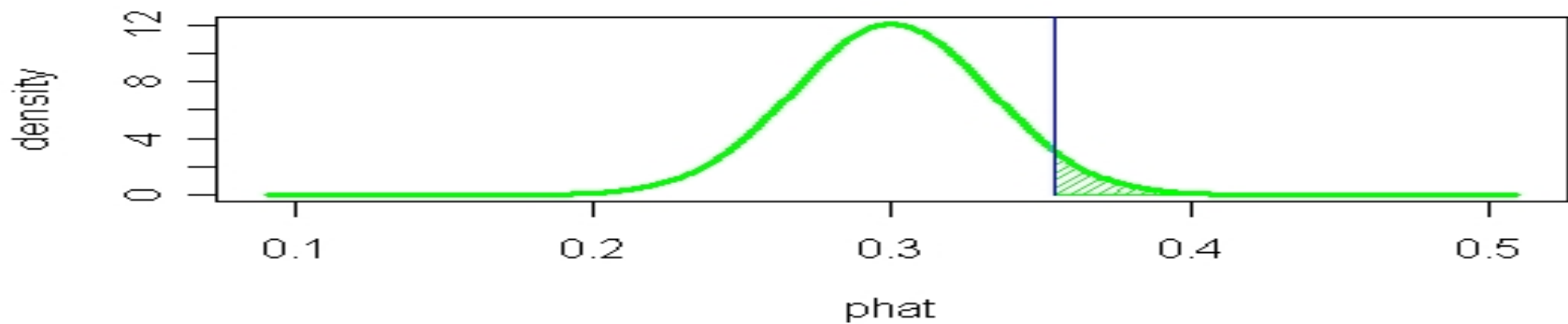
$$H_0 : p = p_0, H_1 : p < p_0$$



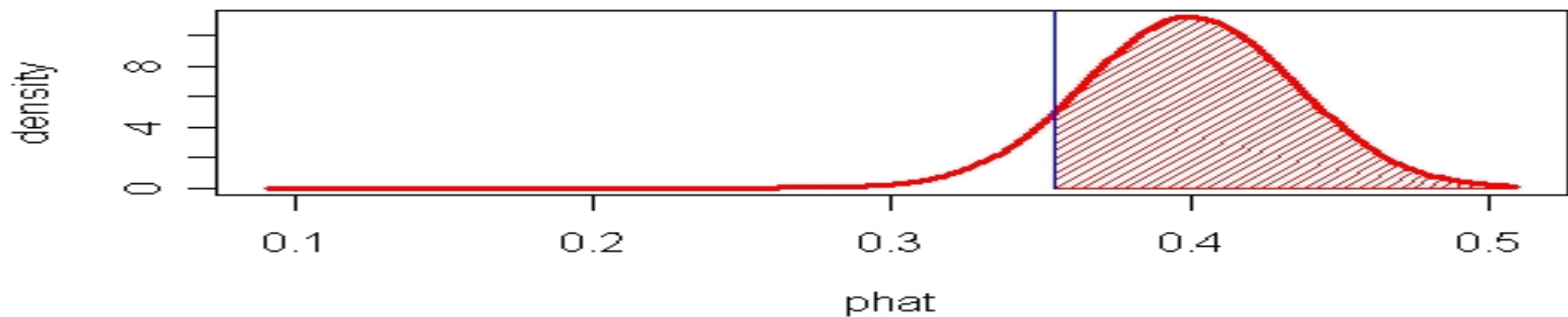
Equate the α percentile of the null to the POW percentile of the alternative



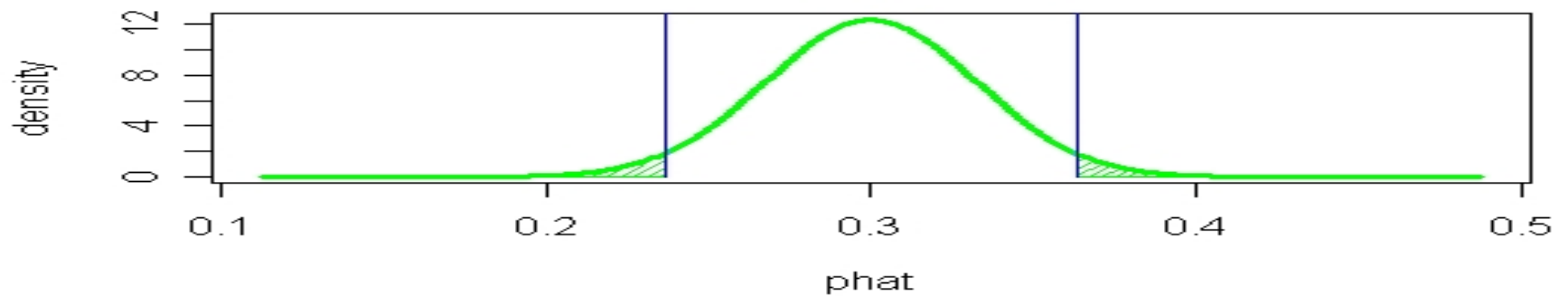
$$H_0 : p = p_0, H_1 : p > p_0$$



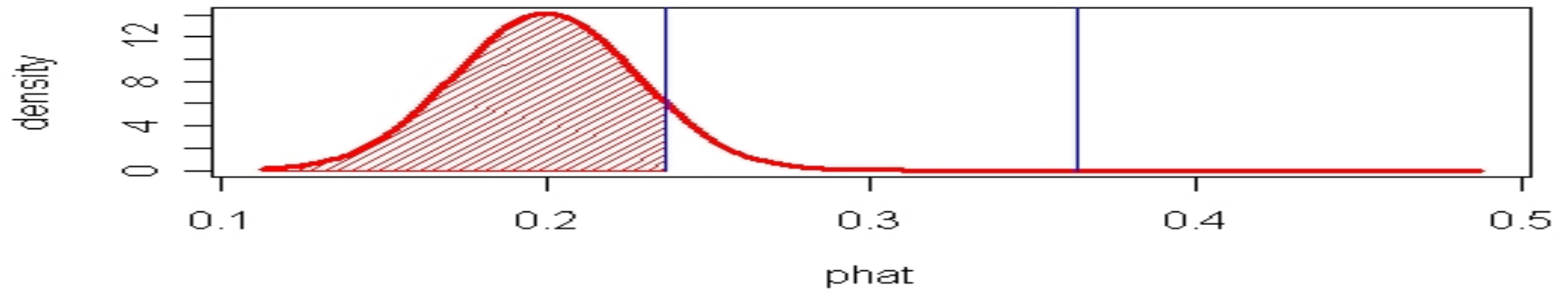
Equate the $1-\alpha$ percentile of the null to the $1-\text{POW}$ percentile of the alternative



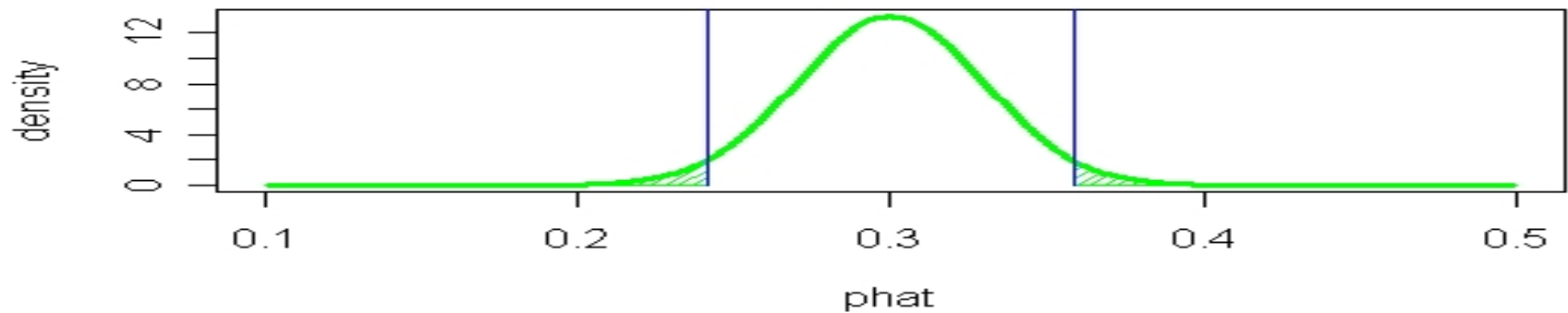
$H_0 : p=p_0, H_1 : p \neq p_0, \text{ with } p_1 < p_0$



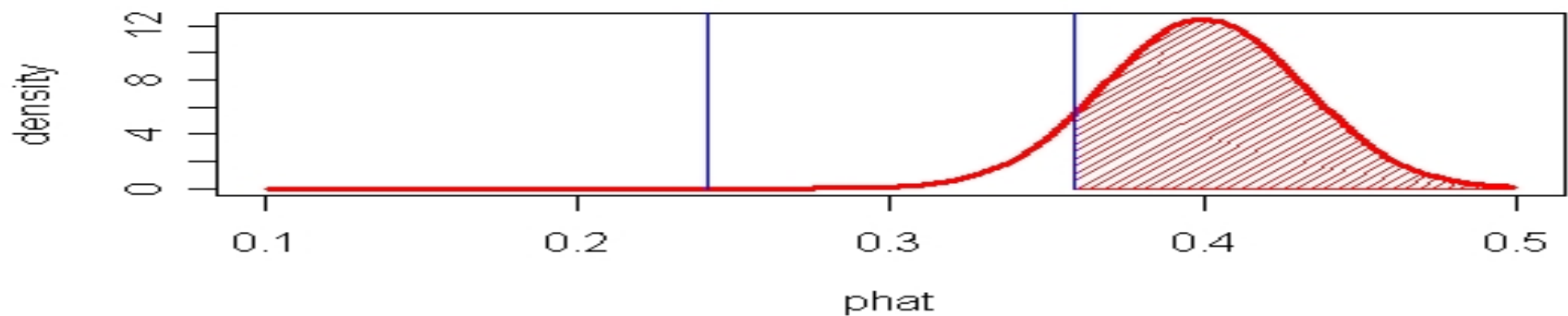
Equate the $\alpha/2$ percentile of the null to the POW percentile of the alternative



$H_0 : p=p_0, H_1 : p \neq p_0, \text{ with } p_1 > p_0$



Equate the $1-(\alpha/2)$ percentile of the null to the 1-POW percentile of the alternative



A common thread

- Every situation requires equating a percentile of the null distribution to a percentile of the alternative distribution.
- Let z_0 be the z-score for the percentile required for the null distribution, and let z_1 be the z-score for the percentile required for the alternative distribution.
- Note the standard deviation for the null distribution is $\sqrt{p_0(1-p_0)/n} = \sqrt{p_0(1-p_0)}/\sqrt{n}$. Let $s_0 = \sqrt{p_0(1-p_0)}$, thus the standard deviation of the null is s_0/\sqrt{n} .
- Similarly, define $s_1 = \sqrt{p_1(1-p_1)}$, so the standard deviation of the alternative is s_1/\sqrt{n} .

General Formula

- We need to equate a percentile of the null distribution to a percentile of the alternative distribution.
- For the null, we need $p_0 + z_0 (s_0/\sqrt{n})$
- For the alternative, we need $p_1 + z_1 (s_1/\sqrt{n})$
- Thus we need to solve the equation below for n

$$p_0 + z_0 \left(\frac{s_0}{\sqrt{n}} \right) = p_1 + z_1 \left(\frac{s_1}{\sqrt{n}} \right)$$

A general formula

$$p_0 + z_0 \left(\frac{s_0}{\sqrt{n}} \right) = p_1 + z_1 \left(\frac{s_1}{\sqrt{n}} \right)$$

$$p_0 - p_1 = \frac{z_1 s_1 - z_0 s_0}{\sqrt{n}}$$

$$n = \left(\frac{z_1 s_1 - z_0 s_0}{p_0 - p_1} \right)^2$$

Choose the next larger integer.

Sample Size: Test for a Proportion

	One-Sided Tests		Two-Sided Test
Null Hypothesis	$H_0 : p = p_0$		
Research Hypothesis	$H_1 : p < p_0$	$H_1 : p > p_0$	$H_1 : p \neq p_0$
Sample Size	$\left(\frac{z_{1-\beta}S_1 + z_{1-\alpha}S_0}{p_0 - p_1} \right)^2$		$\left(\frac{z_{1-\beta}S_1 + z_{1-\alpha/2}S_0}{p_0 - p_1} \right)^2$

$z_p = p$ th quantile of standard normal

Sample Size: Test for a Mean

	One-Sided Tests		Two-Sided Test
Null Hypothesis	$H_0 : \mu = \mu_0$		
Research Hypothesis	$H_1 : \mu < \mu_0$	$H_1 : \mu > \mu_0$	$H_1 : \mu \neq \mu_0$
Sample Size	$\left(\sigma \frac{z_{1-\beta} + z_{1-\alpha}}{\mu_0 - \mu_1} \right)^2$		$\left(\sigma \frac{z_{1-\beta} + z_{1-\alpha/2}}{\mu_0 - \mu_1} \right)^2$

$z_p = p$ th quantile of standard normal

Example

- We want to test $H_0 : p=0.7$ against $H_1 : p>0.7$.
- We want to use $\alpha=0.01$ and achieve 95% power when $p=0.80$.
- What is the minimum required sample size n ?
- We want to equate the $1-\alpha=0.99$ percentile of the null distribution to the $1-\text{POWER}=0.05$ percentile of the alternative distribution.
- For the formula: $z_{1-\alpha}=2.33$ and $z_{1-\beta}=1.64$.
- Also $s_0=\text{sqrt}(p_0(1-p_0))=\text{sqrt}(0.7*0.3)=0.4583$ and $s_1=\text{sqrt}(p_1(1-p_1))=\text{sqrt}(0.8*0.2)=0.4000$.

Example of using the formula

$$n = \left(\frac{z_{1-\beta} s_1 + z_{1-\alpha} s_0}{p_0 - p_1} \right)^2$$

$$n = \left(\frac{(1.64)(0.4000) + (2.33)(0.4583)}{0.7 - 0.8} \right)^2 = 297.2$$

Thus n must be at least 298.

Significance Test for a Mean

Example

- The mean score for all high school seniors taking a college entrance exam equals 500.
- A study is conducted to see whether a different mean applies to those students born in a foreign country.
- For a random sample of 100 of such students, the mean and standard deviation on this exam equal 508 and 100.
- Q: Is mean significantly different from 500 for international students?

Significance Test for a Mean

Assumptions

- What type of data?
 - *Quantitative*
- What is the population distribution?
 - *No special assumptions.*
 - *The test refers to the population mean of the quantitative variable.*
- Which sampling method has been used?
 - *Random sampling*
- What is the sample size?
 - *Minimum sample size of $n=30$ to use Central Limit Theorem with estimated standard deviation*

Significance Test for a Mean

Hypotheses

- The null hypothesis has the form $H_0 : \mu = \mu_0$ where μ_0 is an a priori (before taking the sample) specified number like 0 or 5.3 or 500
- The most common alternative hypothesis is
$$H_1 : \mu \neq \mu_0$$
- This is called a two-sided hypothesis, since it includes values falling above and below the null hypothesis

Significance Test for a Mean

Test Statistic

- The hypothesis is about the population mean
- So, a natural test statistic would be the sample mean
- The sample mean has, for sample size of at least $n=25$, an approximately normal sampling distribution
- The parameters of the sampling distribution are, under the null hypothesis,
 - Mean = μ_0 (that is, the sampling distribution is centered around the hypothesized mean)
 - Standard error = $\frac{\sigma}{\sqrt{n}}$, estimated by $\frac{s}{\sqrt{n}}$

Significance Test for a Mean

Test Statistic

- Then, the z-score has a standard normal distribution

$$z = \frac{\bar{Y} - \mu_0}{s / \sqrt{n}}$$

- The z-score measures how many estimated standard errors the sample mean falls from the hypothesized population mean
- The farther the sample mean falls from μ_0 the larger the absolute value of the z test statistic, and the stronger the evidence against the null hypothesis

Significance Test for a Mean

p-Value

- The p -value has the advantage that different test results from different tests can be compared: The p -value is always a number between 0 and 1
- It is the probability that a standard normal distribution takes values more extreme than the observed z score
- The smaller the p -value is, the stronger is the evidence against the null hypothesis and in favor of the alternative hypothesis
- Round p -value to two or three significant digits

Example

- The mean score for all high school seniors taking a college entrance exam equals 500.
 - A study is conducted to see whether a different mean applies to those students born in a foreign country.
 - For a random sample of 100 of such students, the mean and standard deviation on this exam equal 508 and 100.
1. Set up hypotheses for a significance test.
 2. Compute the test statistic.
 3. Report the P -value, and interpret.
 4. Can you conclude that the population mean for students born in a foreign country equals 500? Why or why not?
 5. Make a decision about H_0 , using $\alpha=0.05$
 6. Construct a 95% confidence interval for μ .