

STA 321

Spring 2014

Lecture 22

Thursday, April 17

➤ **Analysis of Variance**

Analysis of Variance

- Earlier
 - Comparing Two Samples (Binary, Ordinal, Quantitative)
 - Example: Compare mean annual income of men and women
- Now
 - Several samples, Quantitative variable
 - Comparing Several Means
 - Example: Compare mean statistics exam score for students from several different departments/colleges

Analysis of Variance

- Short: ANOVA
- Developed by Sir R.A. Fisher in the 1920s for agricultural data
- Uses the F -distribution (named after Fisher)
- Goal: Detect evidence of differences between population means

ANOVA Notation

- Number of groups: g
- Means of the response variables for the g populations: $\mu_1, \mu_2, \dots, \mu_g$
- Null hypothesis: All the means are equal, that is, $H_0 : \mu_1 = \mu_2 = \dots = \mu_g$
- Group sample sizes: n_1, n_2, \dots, n_g
ANOVA works best when the sample sizes are equal (balanced design)
- Total sample size: $N = n_1 + n_2 + \dots + n_g$
- Sample Means: $\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_g$
- Sample standard deviations: s_1, s_2, \dots, s_g

ANOVA Assumptions

- Independent random samples are selected from the g populations (***very important***)
- Response is truly quantitative (***very important***)
- The standard deviations of the population distributions for the g groups are equal, denoted by σ (***if this is not the case, there is a statistical solution: variance stabilizing transformations***)
- The population distributions of the response variable are normal for each of the g groups (***important for small sample sizes and small number of samples***)

Variability Between and Within Groups

- ANOVA compares two types of variability
 - Variability of the sample observations about their separate means (*within-group variability*)
 - Variability of the sample means from the different groups about the overall mean (*between-group variability*)

Variability Between and Within Groups

- The greater the variability between sample means and the smaller the variability within each group of observations, the stronger the evidence that the null hypothesis of equal means is false
- The test statistic is the ratio of two variance estimates:
 - Between-groups estimate and
 - Within-groups estimate

Within-Groups Estimate of Variance

Technical Details

- For each of the g groups, a variance estimate can be calculated
- Construct a ***pooled variance estimate*** by adding up weighted group variance estimates
- The weights are the degrees of freedom for each group: $n_i - 1$
- Divide by the total degrees of freedom: $N - g$

Within-Groups Estimate of Variance

Technical Details

- This estimate is a weighted average of the separate sample variances, with greater weight given to larger samples
- It is unbiased and efficient
- Mathematical Formula:

$$\hat{\sigma}^2 = \frac{WSS}{N - g} = \frac{SSE}{N - g} = MSE$$
$$= \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_g - 1)s_g^2}{N - g}$$

Between-Groups Estimate of Variance

- This estimate is based on the variability between each sample mean and the overall mean (from all samples together)
- Under the null hypothesis, it is unbiased
- Mathematical Formula:

$$\frac{BSS}{g - 1} = MSH$$
$$= \frac{n_1(\bar{Y}_{1.} - \bar{Y}_{..})^2 + n_2(\bar{Y}_{2.} - \bar{Y}_{..})^2 + \cdots + n_g(\bar{Y}_{g.} - \bar{Y}_{..})^2}{g - 1}$$

$$\text{where } \bar{Y}_{..} = \frac{1}{N} \sum Y_{ij}$$

F Test Statistic

The test statistic for the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_g$$

is the ratio of the two variance estimates:

$$F = \frac{\text{Between-Groups Estimate}}{\text{Within-Groups Estimate}}$$
$$= \frac{BSS / (g - 1)}{WSS / (N - g)} = \frac{MSH}{MSE}$$

F Test Statistic

- It is called the ***analysis of variance F statistic*** or ***ANOVA F statistic***
- If the null hypothesis is true, its sampling distribution is the *F* distribution with degrees of freedom $df_1 = g - 1$ and $df_2 = N - g$
- The P-value is the right-tail probability that the *F* test statistic takes a value at least as large as the observed *F* value
- The larger the *F* test statistic, the smaller the P-value

F Test Statistic

- If the null hypothesis is true, numerator and denominator are both unbiased estimates of the same quantity
- We expect the ratio to be around 1
- If the null hypothesis is not true, the numerator will be larger, and the test statistic takes larger values
- We reject the null hypothesis when the values are too large
- What is too large? Sampling distribution

Example (contd.)

- The quiz scores in the beginning French course are given in the table
- Calculate the F statistic for the quiz score example
- What is the P-value?
- [F distribution online tool](#)

Group A	Group B	Group C
4	1	9
6	5	10
8		5

ANOVA Table (SAS Output)

- Statistical software displays the results of ANOVA *F* tests in a table called ***ANOVA table***

The GLM Procedure
Class Level Information

Class	Levels	Values
group	3	A B C

Number of observations 8

Dependent Variable: score

Sum of

Source	DF	Squares	Mean Square	F Value	Pr > F
Model	2	30.00000000	15.00000000	2.50	0.1768
Error	5	30.00000000	6.00000000		
Corrected Total	7	60.00000000			

Level of group	N	Mean	Std Dev
A	3	6.00000000	2.00000000
B	2	3.00000000	2.82842712
C	3	8.00000000	2.64575131

Another Example

- Three materials for making artificial teeth are compared with regard to hardness.
- The materials are Endura, Duradent, and Duracross.
- Six pairs of teeth are tested for each material.
- The response variable is the Vickers microhardness of the occlusal surfaces, measured with a load of 50 g and a loading time of 30 sec.

Example: Hardness of Artificial Teeth (contd.)

- Data table, with sample means and standard deviations

	Endura	Duradent	Duracross
Hardness	27.1 27.6 28 28.5 27.3 26.7	23.9 24.5 23.9 24.4 22.9 24.5	44.9 37.9 40.4 38.5 40.4 35.7
<i>Sample Mean</i>	<i>27.53</i>	<i>24.02</i>	<i>39.63</i>
<i>Sample Standard Deviation</i>	<i>0.65</i>	<i>0.61</i>	<i>3.12</i>

ANOVA Table (from SAS)

The SAS System
The GLM Procedure

19:52 Monday, March 31, 2008

Class Level Information

Class	Levels	Values
MATERIAL	3	Duracros Duradent Endura
Number of Observations Read		18
Number of Observations Used		18

Dependent Variable: HARDNESS

	Sum of				
Source	DF	Squares	Mean Square	F Value	Pr > F
Model	2	805.3144444	402.6572222	114.71	<.0001
Error	15	52.6550000	3.5103333		
Corrected Total	17	857.9694444			

Interpretation

- The null hypothesis for the ANOVA is

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_g$$

- The P-value from the ANOVA table is <0.0001
- At 5% level, there is sufficient evidence against the null hypothesis
- So we can conclude that not all the population means are equal

Interpretation, contd.

- However, the conclusion of the test does not specify which means are different or how different they are
- More detailed inference is necessary to determine the nature of the differences

Multiple Comparisons of Means

- Confidence intervals are usually more informative than test results
- In practice, we would be interested in estimates of the population means and confidence intervals for their differences
- Compare groups A vs. B, A vs. C, B vs. C
- We can also perform pairwise t-tests with some correction like Bonferroni procedure
- “post-hoc (*after this*) analysis”

Multiple Comparisons Using SAS

```
data teeth;
input hardness material$;
cards;
27.1 Endura
27.6 Endura
28      Endura
28.5 Endura
27.3 Endura
26.7 Endura
23.9 Duradent
24.5 Duradent
23.9 Duradent
24.4 Duradent
22.9 Duradent
24.5 Duradent
44.9 Duracross
37.9 Duracross
40.4 Duracross
38.5 Duracross
40.4 Duracross
35.7 Duracross
;
```

```
proc glm data=teeth;
class material;
model hardness=material;
means material/bon
alpha=0.05;
run;
```

SAS Output

Bonferroni (Dunn) t Tests for *hardness*

NOTE: This test controls the **Type I *experimentwise* error rate**

Alpha ***0.05***

Error Degrees of Freedom 15

Error Mean Square 3.510333

Critical Value of t 2.69374

Minimum Significant Difference ***2.9139***

Means with the same letter are not significantly different.

Bon	Grouping	Mean	N	type
<i>A</i>	<i>39.633</i>	6	Duracros	
<i>B</i>	<i>27.533</i>	6	Endura	
<i>C</i>	<i>24.017</i>	6	Duradent	

Interpretation

- ANOVA F test:
 - The population means are not all the same
- Pairwise comparisons:
 - Duracross is significantly harder than Endura and than Duradent
 - Endura is significantly harder than Duradent

Summary

- Use ANOVA to check whether population means for g groups are identical
- Quantitative response,
qualitative explanatory variable (group)
- If (and ONLY IF) there is enough evidence that the population means are not all identical, perform pairwise comparisons to find out which pairs are significantly different

Quiz I

When testing the null hypothesis using the confidence interval estimate of the difference between two means, one would reject the null hypothesis (two sided test) when

- (a) the upper confidence limit is greater than zero
- (b) the confidence interval does not include zero
- (c) the confidence interval includes zero
- (d) the lower confidence limit is less than zero

Quiz II

- We have created a 90% confidence interval for μ with the result (25, 32). What conclusion will we make if we test $H_0: \mu = 28$ vs. $H_1: \mu \text{ not } = 28$ at $\alpha = 0.10$?
 - a. Reject H_0 in favor of H_1 .
 - b. Accept H_0 in favor of H_1 .
 - c. Fail to reject H_0 in favor of H_1 .
 - d. We cannot tell from the information given.