STA 321 Spring 2015

Lecture 10 *Tuesday, Feb 17*

- **Sampling Distribution (Sec 8.1)**

Recall: Parameters

- A numerical aspect of the population is called a **parameter**.
- Typically, we would like to make statements about parameters, but they are unknown.
- Voting example: If we are trying to predict an election, we want to know which way the ***entire population of voters*** would vote if the election were held today. The population proportion of voters for candidate *A* is a parameter.

Parameters and Statistics

- The proportion of **all** voters who would today vote for *A* (however you define *A*) is a parameter.
- A ***statistic*** is any numerical aspect of the sample.
- We observe the sample, and thus we can calculate the statistic.
- Our goal is to use those known statistics to estimate the unknown population parameters.
- Fortunately, calculated from a good sample or experiment, sample statistics are close to population parameters.
- This forms the basis of **statistical inference**.

Sampling Distributions

- For the probability theory to work, your samples need to be drawn randomly from the population;
- Recall: "Simple random sample" means that every sample has the same probability of being chosen.
- Unfortunately, random sample will give different results each time – because of sampling variation.
- Fortunately, however, probability theory allows us to conclude that there is a ***predictable pattern of variation*** among the samples.

Simple Example 1

- Suppose we have a population of 20 people, 12 of which will vote for *X* and 8 will vote for *Y*. We sample 5 people at random.
- The population will usually be bigger in practice, as will our sample, this is just for illustration.
- Also, in practice, we will obviously not know that 12 (=60%) will vote for *X* and 8 (=40%) will vote for *Y*.
- Label the people A, B, C, D, …, T. We could sample ABCDE, or ABCDF, or ABCDG, or DNORT, or any of the other 15,504 possibilities.

Simple Example 1, contd.

- Each of the 15,504 possible samples of 5 people are equally likely. We don't know which one we will get.
- Probability Theory (not required in STA 321) allows us to determine that 56 of these possible samples have 0 "yes" people, 840 have 1 "yes" person, 3696 have 2 "yes" people, and so forth.

Simple Example 1, contd.

# (%) "yes" Responses	Number of possible samples	Proportion of possible samples
0 (0% yes)	56	0.36%
1 (20% yes)	840	5.42%
2 (40% yes)	3696	23.84%
3 (60% yes)	**6160**	**39.73%**
4 (80% yes)	3960	25.54%
5 (100% yes)	792	5.11%
Total	15504	100%

Simple Example 1, concluded

- Is your sample proportion guaranteed to be 0.60, exactly equal to your population proportion? No, but there is about a 40% chance it is.
- There is close to a 90% chance that the sample proportion will be within 0.2 of the population proportion.
- Thus, there is a high likelihood that a sample statistic calculated from a random sample will be close to the true (usually unknown) population proportion.
- With more realistic population and sample sizes, there is an even greater chance that the sample statistic will be close to the population parameter.

Example 2

- Suppose there are 10,000 students on a campus.
- We want to know the average height, but can not measure all 10,000.
- Instead, we sample 100 individuals and measure those.
- Thus, we get to see just one of the *large* number of possible samples of 100 people out of 10,000.
- Fortunately, probability theory still says that our sample average height should be CLOSE to the population average height.
- How close… probability theory will tell us that, too…but we'll have to wait to find out.

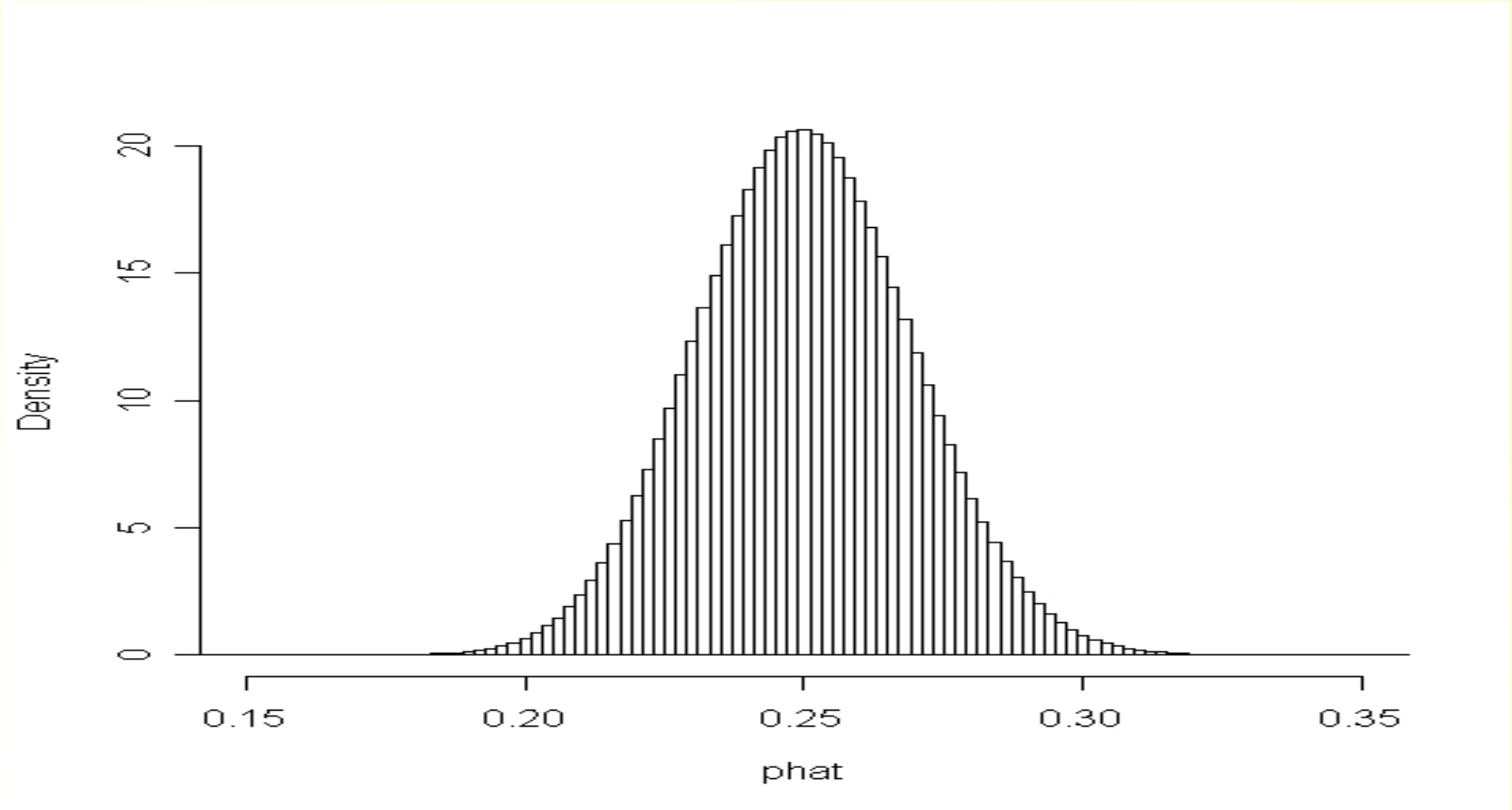
Example 3

- We are interesting in determining what proportion of a population visits a doctor at least once a year.
- Our population contains 100,000 individuals. Unknown to us, 25,000 visit a doctor at least once a year while 75,000 do not.
- We decide to sample 500 at random and determine whether those individuals visit a doctor at least once a year (termed a success), as opposed to those who do not visit a doctor at least once a year (termed a failure).

- Note our population parameter is p=0.25 (25,000 out of 100,000). This is typically unknown.
- Our sample of 500 might yield 130 successes, resulting in a sample proportion phat=0.260, or our sample of 500 might yield 122 successes, resulting phat=0.244.
- Because our sample is (and should be!) random, so we are not quite sure what will happen in any *single* sample.
- Again, however, out of the *very many* possible samples, a very large proportion of them have sample proportions close to the true proportion p=0.25.

- It turns out there are over 10^1365 (a one with 1365 zeroes after it) ways to pick 500 people out of 100,000 people. Your sample will be ONE of those many possible samples.
- It is still possible to figure out precisely how many of these samples contain 0 (=0%) successes, 1 (=0.2%) success, 2 (=0.4%) successes, and so on up to 500 (=100%) successes.

Graph of sample proportions for all possible samples for selecting 500 people from a population with 25000 successes and 75000 failures *(sampling distribution)*.

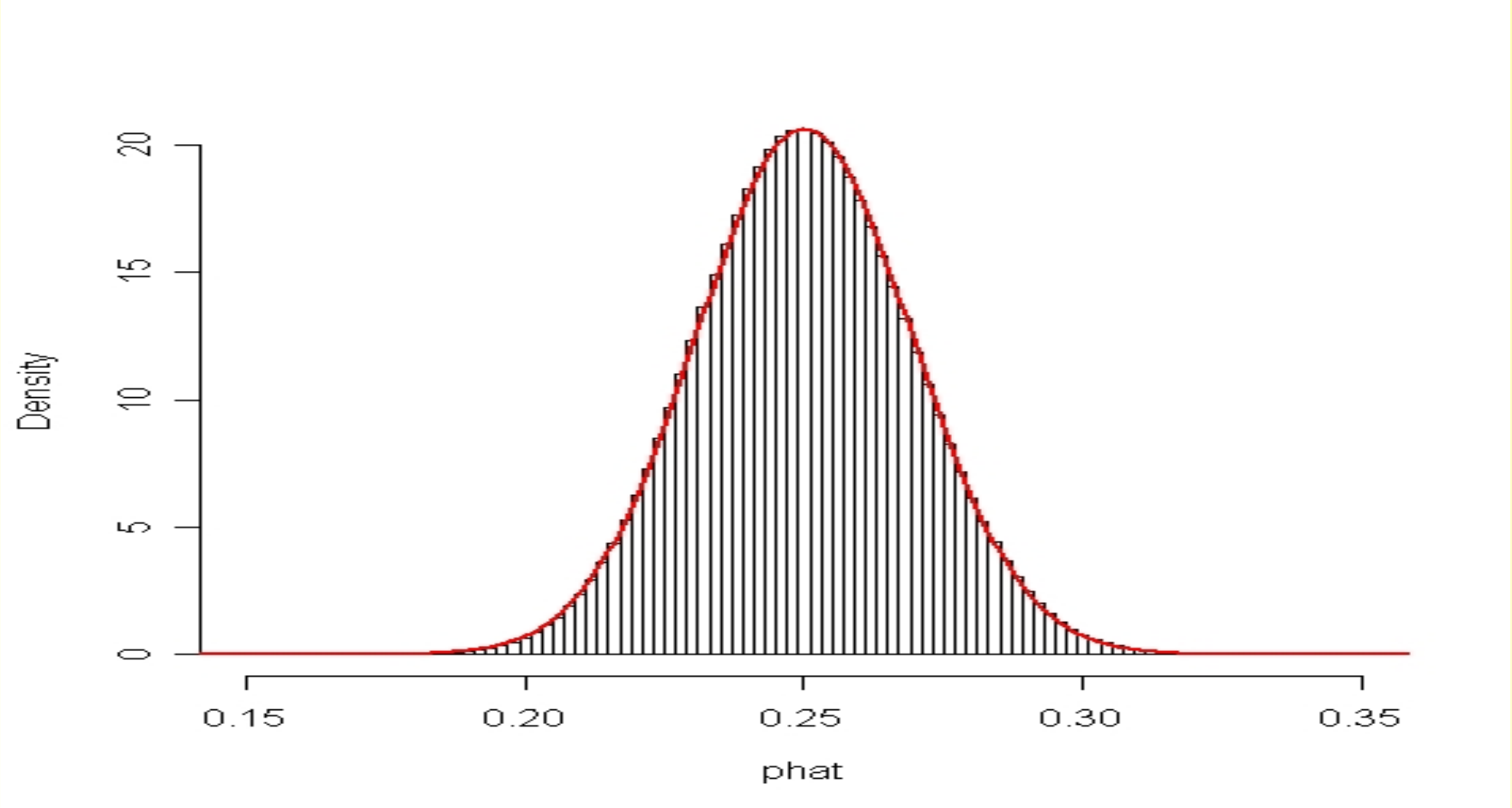


Hm?

- That looks like a bell curve.
- In fact, it looks suspiciously like a bell curve with mean μ=0.25 (that is where the peak is).
- And the standard deviation is (less obvious, but true)

  sqrt(p(1-p)/n) = sqrt(0.25*0.75/500) = 0.0194
- The next graph combines the histogram of sample proportions with the true bell curve with mean =0.25 and standard deviation = 0.0194.

Graph of sample proportions for all possible samples for selecting 500 people from a population with 25000 successes and 75000 failures, overlayed with a perfect normal curve.



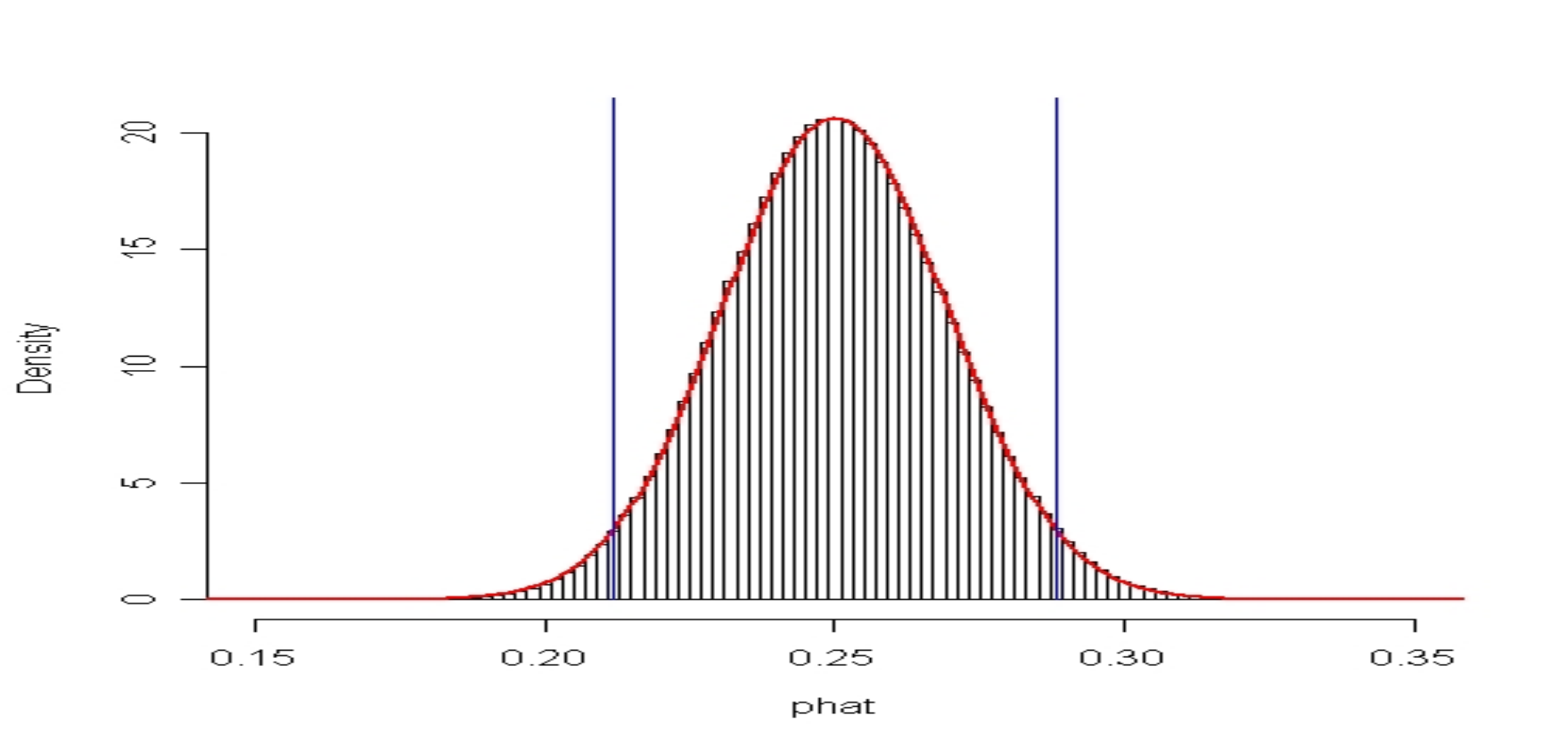
Review

- We cannot tell what will happen in any given individual sample (just as we can not predict a single coin flip in advance).
- We CAN tell a lot about the pattern of variation amongst many samples (just as we can predict that if you flip the coin a lot, you will get about 50% heads and 50% tails).
- In our doctor example, we found that the pattern of variation of the sample proportions, called the ***sampling distribution***, followed a normal distribution.

Useful Consequences

- In our Example 3 (doctor visits), we know the sampling distribution of the sample proportion of successes is N(0.25,0.0194).
- Recall the 68-95-99.7 rule. We know there is about 95% probability that the sample proportion will be between 2 standard deviations (2*0.0194=0.0388) of the population proportion.
- There is a 99.7% chance the sample proportion will be within 3 standard deviations (0.0582) of the population proportion.

Empirical Rule: About 95% of our observations should fall between the blue lines



- In actuality, we have 95.5%.

Sampling Distributions for Proportions

- Suppose we have a population of size N consisting of M successes and N-M failures.
- We sample a group of n people at random.
- Suppose further that
  - n/N is small (rule of thumb: less than 5%)
  - n is not small (rule of thumb: n>25)
  - M/N=p is not too close to 0 or 1 (rule of thumb: 0.05<p<0.95).
- Then the sampling distribution of the sample proportion is
  - **normal**
  - with **mean M/N=p** (the population proportion)
  - and **standard deviation sqrt(p(1-p)/n**).
- *Why this is true is beyond the scope of this course. It is because of a beautiful mathematical theorem:* ***Central Limit Theorem.***

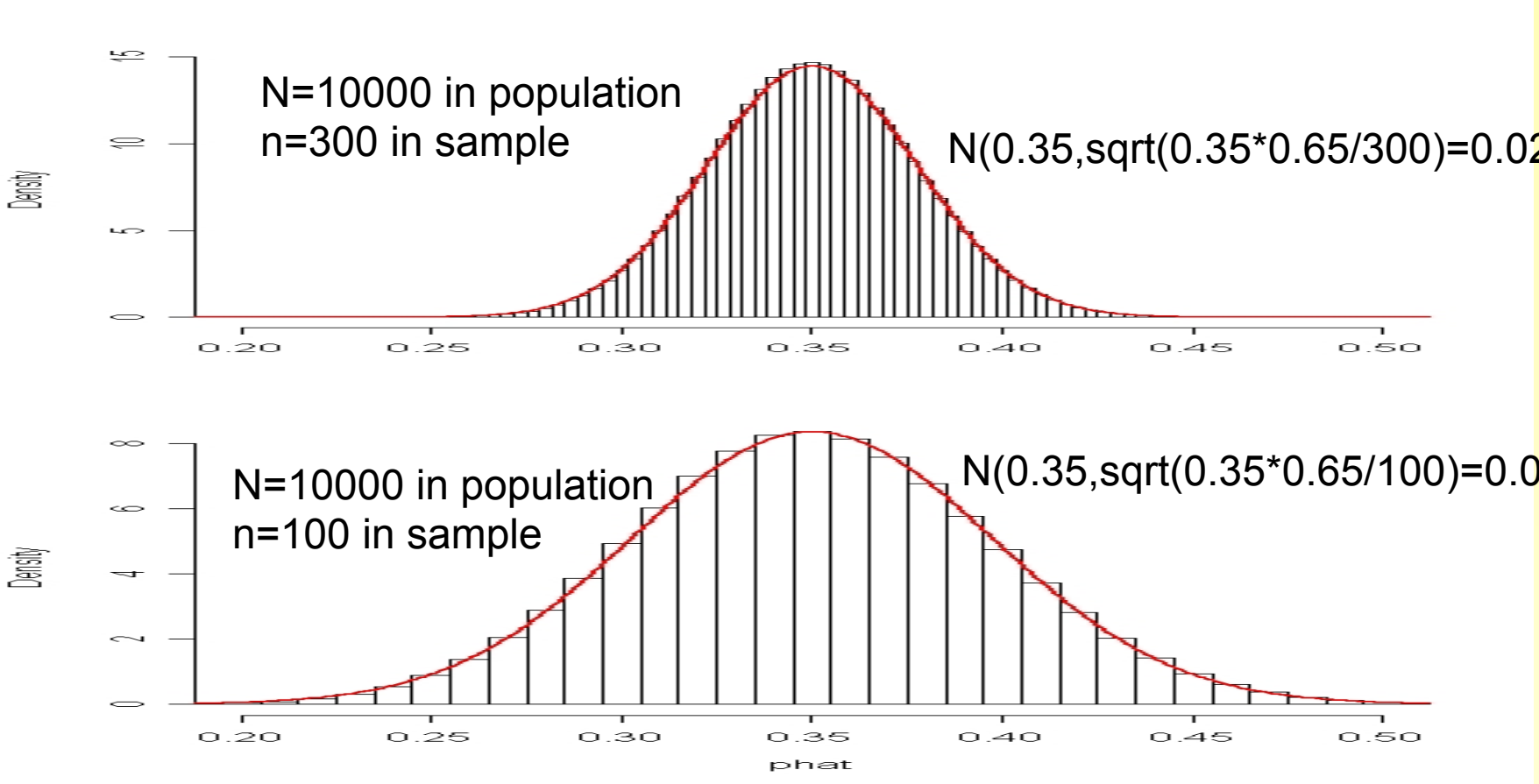
In Practice

- Unfortunately, we typically only get to draw one sample. How do you know if you got one of the samples that fall in the middle 95% (closer to the true proportion) as opposed to the outer 5% (farther from the true proportion)?
- Answer – really, you don't.
- But it's more likely you're in the 95% group than the 5% group.
- Want to be more sure?
- Construct a 99% group instead of a 1% group, then the odds are even more in your favor.

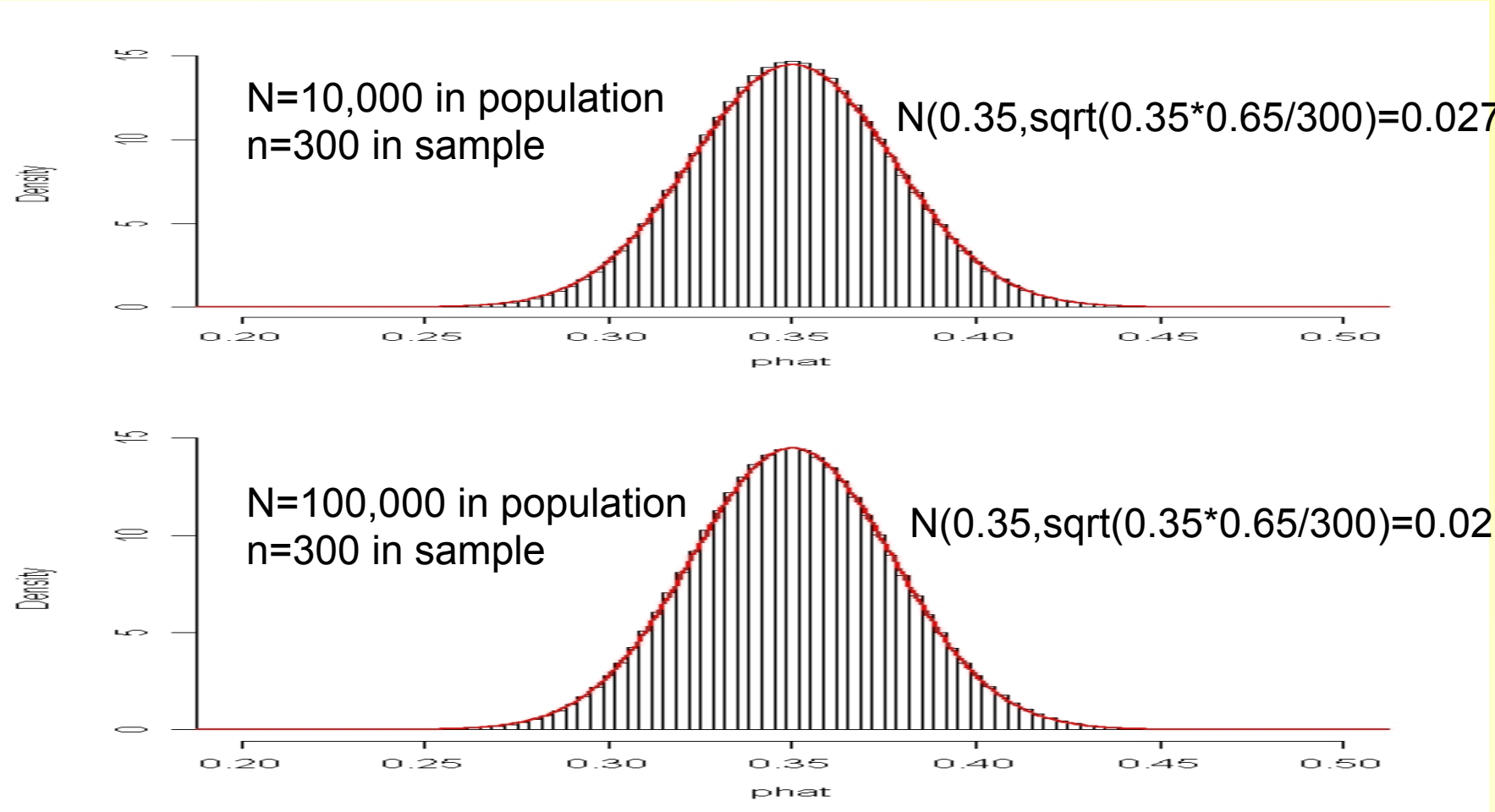
What Matters, What Doesn't

- The center of the sampling distribution is the true proportion p.
- On average, p-hat is centered around p.
- The sample size appears in the standard deviation sqrt(p(1-p)/n).
- The bigger the sample size, the smaller the standard deviation of p-hat. In other words, the closer p-hat tends to be to p.
- The population size does NOT matter.
- As long as you are sampling less than 1 in 20 people, it does not matter whether it is 1 of every 2000 or 1 of every 2 million.

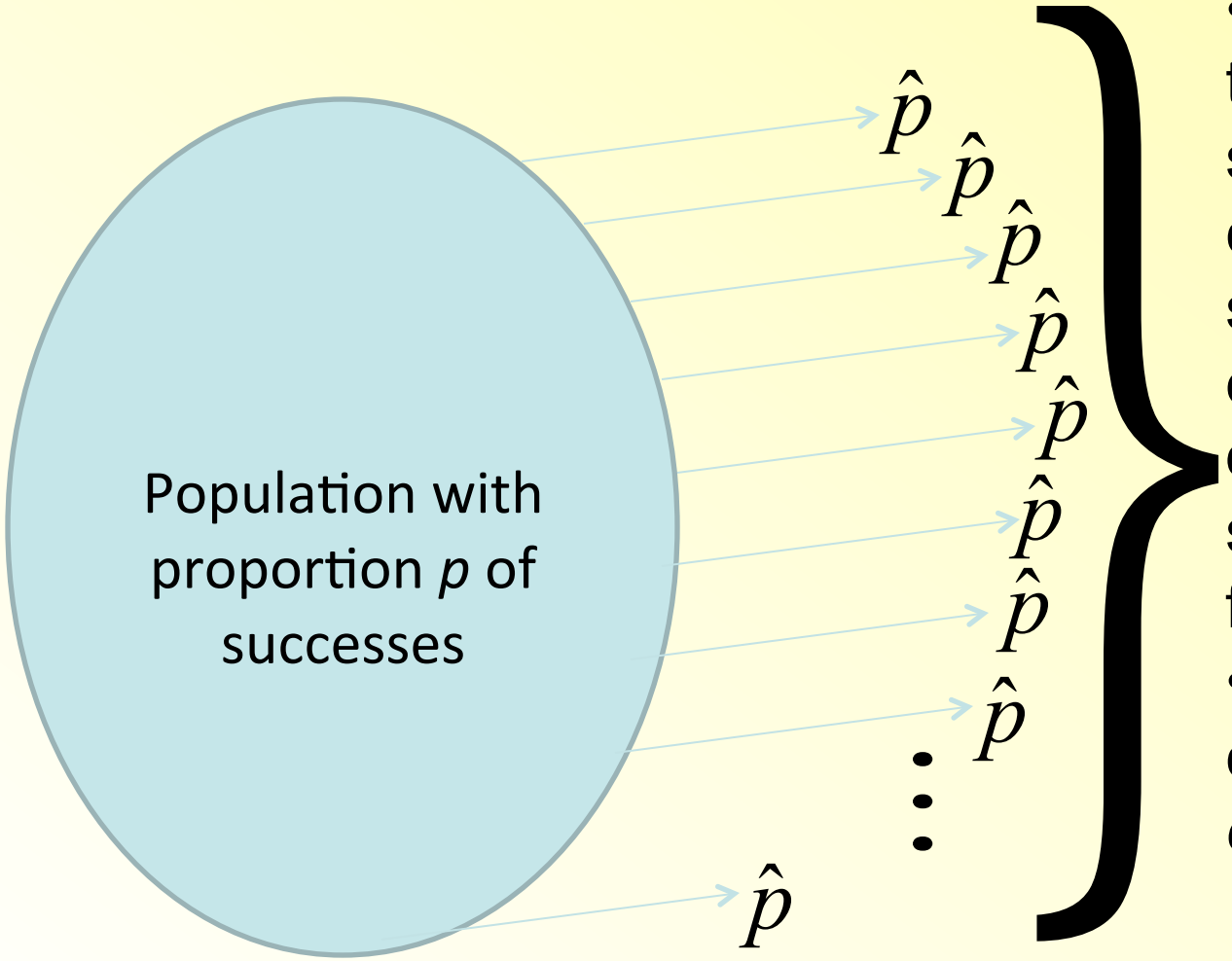
Population Size N=10000, 35% Successes Comparing n=300 to n=100



Sample Size n=300, 35% Successes Comparing N=10000 to N=100000



Summary: Sampling Distribution



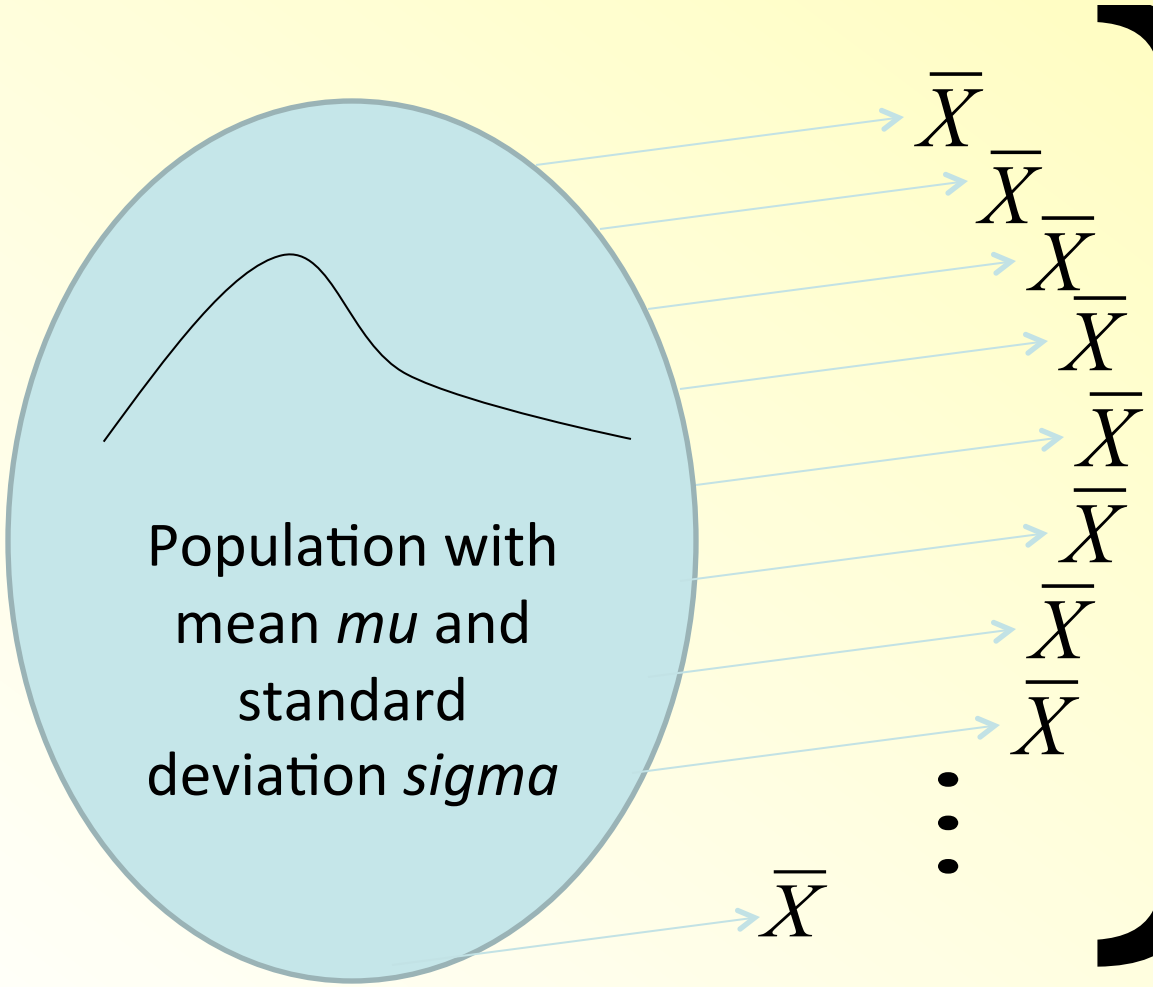
- If you repeatedly take random samples and calculate the sample proportion each time, the distribution of the sample proportions follows a pattern
- This pattern is called the *sampling distribution of p-hat*

Properties of the Sampling Distribution

- Expected Value of the $\hat{p}$'s: p.
- Standard deviation of the $\hat{p}$'s: $\sqrt{\frac{p(1-p)}{n}}$

  also called the *standard error* of $\hat{p}$
- ***Central Limit Theorem***: As the sample size increases, the distribution of the $\hat{p}$'s gets closer and closer to the normal.

Sampling Distribution of Means



- If you repeatedly take random samples and calculate the sample mean each time, the distribution of the sample means follows a pattern
- This pattern is the *sampling distribution of X-bar*

Properties of the Sampling Distribution

- Expected Value of the $\bar{X}$'s: μ.

- Standard deviation of the $\bar{X}$'s: $\frac{\sigma}{\sqrt{n}}$

also called the *standard error* of $\bar{X}$

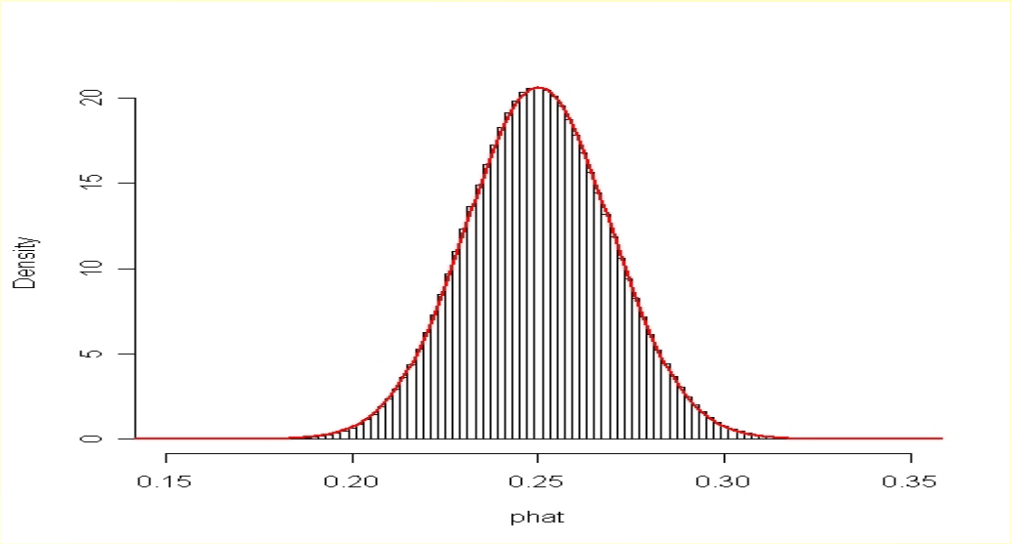
For N/n<20, use a finite population correction factor for the standard deviation: $\sqrt{\frac{N-n}{N-1}}$

- ***Central Limit Theorem***: As the sample size increases, the distribution of the $\bar{X}$'s gets closer and closer to a normal curve.

Summary: Sampling Distribution

- We cannot tell what will happen in any given individual sample.
- We CAN tell a lot about the pattern of variation amongst many samples.

Graph of sample proportions for all possible samples for selecting 500 people from a population with 25000 successes and 75000 failures, overlaid with a perfect normal curve.



Summary: Population, Sample, and Sampling Distribution

- Population
  - Total set of all subjects of interest
  - Can be described by (unknown) parameters
  - Want to make inference about its parameters
- Sample
  - Data that we observe
  - We describe it, using descriptive statistics
  - For large *n*, the sample resembles the population
- Sampling Distribution
  - Probability distribution of a statistic (for example, sample mean, sample proportion)
  - Used to determine the probability that a statistic falls within a certain distance of the population parameter
  - For large *n*, the sampling distribution (of sample mean, sample proportion) looks more and more like a normal distribution

Summary: Central Limit Theorem

- The most important theorem in statistics
- For random sampling, as the sample size *n* grows, the sampling distribution of the sample mean $\overline{Y}$ (and of the sample proportion p-hat) approaches a normal distribution
- Amazing: This is the case even if the population distribution is discrete or highly skewed
  - Online applet 1
  - Online applet 2
- The Central Limit Theorem can be proved mathematically (STA 524)

Central Limit Theorem

- Usually, the sampling distribution of $\bar{Y}$ is approximately normal for sample sizes of at least n=25 (rule of thumb)
- In addition, we know that the parameters of the sampling distribution are mean=mu and standard error= $\frac{\sigma}{\sqrt{n}}$
- For example:

  If the sample size is at least n=25, then with 95% probability, the sample mean falls between

$$\mu - 1.96\frac{\sigma}{\sqrt{n}} \text{ and } \mu + 1.96\frac{\sigma}{\sqrt{n}}$$

(μ = population mean,

σ = population standard deviation)

Calculating *z*-Scores

1. **z-Score for an individual observation**
   - You need to know *Y*, *mu*, and sigma to calculate *z*

$$z = \frac{Y - \mu}{\sigma}$$

2. **z-Score for a sample mean**
   - You need to know *Y-bar*, *mu*, *sigma*, and *n* to calculate *z*

$$z = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}}$$

3. **z-Score for a sample proportion**
   - You need to know *p-hat*, *p*, and *n* to calculate *z*

$$z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

Example I

- For women aged 18-24, systolic blood pressures are normally distributed with mean 114.8 [mm Hg] and standard deviation 13.1 [mm Hg]
- Hypertension is commonly defined as a value above 140. If a woman between 18 and 24 is randomly selected, find the probability that her systolic blood pressure is above 140
- For a sample of 4 women, find the probability that their mean systolic blood pressure is above 140
- *Note that for this problem, we don't actually need the central limit theorem because the variable "blood pressure" has a normal distribution – we don't need to rely on averages.*

Example II

- Analysts think that the length of time people work at a job has a mean of 6.1 years and a standard deviation of 4.3 years.
- Do you expect this distribution to be left-skewed or right-skewed or symmetric? Why?
- Can you calculate the probability that a randomly chosen person spends less than 5 years on his/her job?
- What is the probability that 100 people selected at random spend an average of less than 5 years on their job?

Example III: Acceptance Sampling

- Some companies monitor quality by using a method called acceptance sampling.
- An entire batch of items is rejected if a random sample of a particular size includes more than a specified number of defects.
- Assume that a company buys machine bolts in batches of 5000 and rejects the entire batch if, in a sample of 50, at least 2 defects (4% defects) are found.
- If the supplier manufactures bolts with a defect rate of 10%, what is the probability that a random batch will be rejected? How about the rejection rule "4 out of 100"?
- NB: *When we use the continuous normal distribution to approximate a discrete distribution such as "number of defects", a continuity correction should be made. That is, the single value x is represented by the interval from x-0.5 to x+0.5.*

Multiple Choice Question

The Central Limit Theorem implies that

1. All variables have approximately bell-shaped sample distributions if a random sample contains at least 30 observations
2. Population distributions are normal whenever the population size is large
3. For large random samples, the sampling distribution of $\overline{Y}$ is approximately normal, regardless of the shape of the population distribution
4. The sampling distribution looks more like the population distribution as the sample size increases
5. All of the above

Statistical Inference: Estimation

- Recall: Inferential statistical methods provide predictions about characteristics of a population, based on information in a sample from that population
- For quantitative variables, we usually estimate the population mean (for example, mean household income)
- For qualitative variables, we usually estimate population proportions (for example, proportion of people voting for candidate A)

Two Types of Estimators

- Point Estimate
  - A single number that is the best guess for the parameter
  - For example, the sample mean is usually at good guess for the population mean
- Interval Estimate
  - A range of numbers around the point estimate
  - To give an idea about the precision of the estimator
  - For example, “the proportion of people voting for candidate A is between 67% and 73%”

Point Estimator

- A point estimator of a parameter is a sample statistic that predicts the value of that parameter
- A good estimator is
  - ***Unbiased***: Centered around the true parameter
  - ***Consistent***: Gets closer to the true parameter as the sample size gets larger
  - ***Efficient***: Has a standard error that is as small as possible

Unbiased

- An estimator is unbiased if its sampling distribution is centered around the true parameter
- For example, we know that the mean of the sampling distribution of $\bar{Y}$ equals mu, which is the true population mean
- So, $\bar{Y}$ is an unbiased estimator of mu

Unbiased

- However, for any particular sample, the sample mean $\overline{Y}$ may be smaller or greater than the population mean
- "Unbiased" means that there is no systematic under- or overestimation
- If you repeatedly took samples, then the average of the sample means would converge to the population mean

Biased

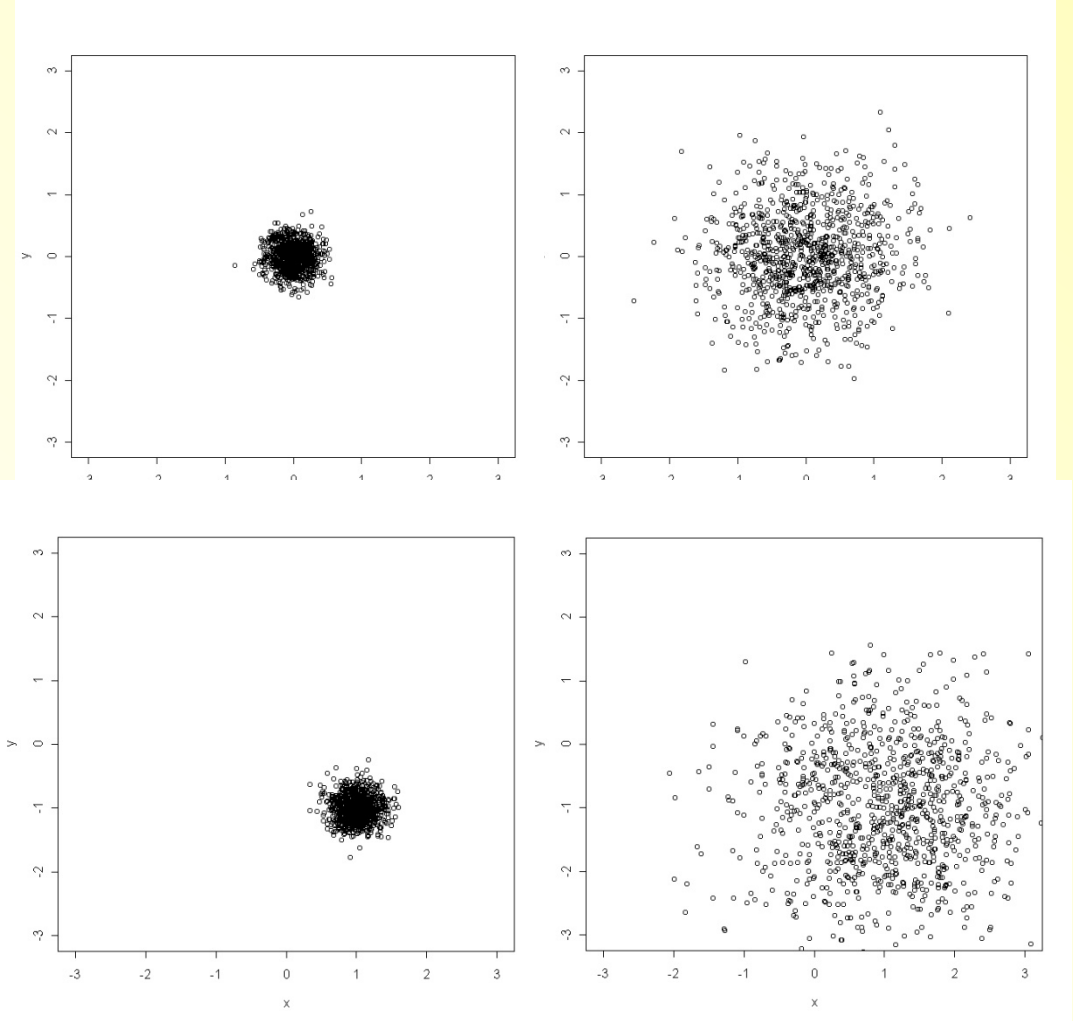
- A biased estimator systematically under- or overestimates the population parameter
- The definition of sample variance and sample standard deviation uses *n-1* instead of *n*, because this makes the variance estimator unbiased
- With *n* in the denominator, it would systematically underestimate the variance

Efficiency

- An estimator is efficient if its standard error is small compared to other estimators
- Such an estimator has high precision
- A good estimator has ***small standard error and small bias*** (or no bias at all)

- The following pictures represent different estimators with different bias and efficiency
- Assume that the true population parameter is the point (0,0) in the middle of the picture

Bias and Efficiency



Note that even an unbiased and efficient estimator does not always hit exactly the population parameter.

But in the long run, it is the best estimator.

Point Estimators of the Mean, Median, and Standard Deviation

- The sample mean is unbiased, consistent, and sometimes relatively efficient
- It is the most efficient estimator when the population distribution is normal (can be proved mathematically)
- The sample median is more efficient for many skewed and "heavy-tailed" distributions
- The sample variance is unbiased when we use *n-1* in the denominator
- It is also consistent (and in some situations relatively efficient)

Example: Three Estimators

- Suppose we want to estimate the proportion of UK students voting for candidate A
- We take a random sample of size $n=100$
- The sample is denoted $Y_1, Y_2, \ldots, Y_n$, where $Y_i=1$ if the ith student in the sample votes for A, $Y_i=0$ otherwise

Example: Three Estimators

- Estimator 1 = the sample mean (sample proportion)
- Estimator 2 = the answer from the first student in the sample (Y_1)
- Estimator 3 = 0.3
- Which estimator is unbiased?
- Which estimator is consistent?
- Which estimator has high precision (small standard error)?