Lecture notes for STA 321 for day 2 by R. Yoshida

1 Sec 6.2: The law of large numbers (part I)

Definition 1. *A random sample is a set of random variables $X_1, \ldots, X_n$ i.i.d. sampled from the sample distribution.*

Remark. The law of large numbers says basically if we have a random sample and if we take the average of the random variables (this is called a sample mean, which we will define later), then the sample mean converges to the expectation of X_i.

In order to prove the law of large numbers we need Markov and Chevychev's inequalities. So first we will show these inequalities.

Theorem 2 (Markov inequality)**.** *Let X be a random variable with $P(X \geq 0) = 1$. The for any $t > 0$, we have*

$$P(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Proof. Note that

□

Theorem 3 (Chevychev's inequality)**.** *Let X be a random variable with $V(X)$ exists. The for any $t > 0$, we have*

$$P(|X - \mathbb{E}| \geq t) \leq \frac{V(X)}{t^2}.$$

Proof. Let

□

Definition 4. *A sample mean $\bar{X}_n$ of n random variables is defined as*

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Theorem 5. *Let $X_1, \dots X_n$ form a random sample from a distribution with its mean μ and its variance σ^2. Then,*

$$\mathbb{E}(\bar{X}_n) = \mu, \;\; V(\bar{X}_n) = \frac{\sigma^2}{n}.$$

Proof. $\mathbb{E}(\bar{X}_n) =$

□

Remark. Using Chevychev's inequality, we can estimate how far the sample mean is from μ by setting

$$P(|\bar{X}_n - \mu| \geq t) \leq \frac{\sigma^2}{nt^2}.$$

Example 6. *Tossing a coin n times with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$, where $0.1 < p < 0.9$ and*

$$X_i = \begin{cases} 1 & \textit{if we observe a head,} \\ 0 & \textit{otherwise.} \end{cases}$$

We want to know the sample size n such that

$$P(p - 0.1 \leq \bar{X}_n \leq p + 0.1)$$

by Chevychev's inequality.

Now we will show some simulations.