**Lecture notes for STA 321 for day 4 by R. Yoshida**

# 1 Sec 6.3: The central limit theorem (part II)

Now we difine the other convergence of a sequence of random variables.

**Definition 1** (Convergence in distribution)**.** *A sequence of random variables $Z_1, Z_2, \ldots$ is said to convergence in distribution to a random variable $Z$ if*

$$\lim_{n\to\infty} F_n(x) = T^*(x)$$

*for all $x \in \mathbb{R}$ where $F_n(x)$ is the cdf of $Z_n$ and $F^*(x)$ is the cdf of $Z$.*

We notate this as

$$Z_n \xrightarrow{d} Z.$$

**Example 2.** *Suppose we have a line (queue) of a service at a cafe shop. Let $X_i$ be the waiting time for the ith person to get a caffee. Suppose $X_1, \ldots, X_n$ be iid random variables with $\mathbb{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then we can estimate the rate of waiting time by $1/\bar{X}_n$. Note that the central limit theorem says $\sqrt{n} \cdot (\bar{X}_n - \mu)/\sigma$ is distributed the statndard normal distribution for large $n$.*

**Theorem 3** (Delta Method)**.** *Suppose $Y_1, \ldots, Y_n$ form a sequence of random variables and let $Y$ be arandom variable. Let $\theta \in \mathbb{R}$ and let $a_1, \ldots a_n$ be a sequence of positive numbers where $a_i \in (0, \infty)$. Let $\alpha$ be a differentiable function with continurous variable such that $\alpha'(\theta) \neq 0$. Then*

$$a_n \left[\alpha(Y_n) - \alpha(\theta)\right] / \alpha'(\theta) \xrightarrow{d} Y.$$

**Corollary 4.** *Let $X_1, \ldots, X_n$ form a sequence of iid random varibales with $\mathbb{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Let $\alpha$ be a differentiable function with continurous variable such that $\alpha'(\mu) \neq 0$. Then,*

$$\frac{\sqrt{n}}{\sigma\alpha'(\mu)} \left[\alpha(\bar{X}_n) - \alpha(\mu)\right] \xrightarrow{d} Z,$$

*where $Z$ is distributed according to the standard normal distribution.*

*Proof.* Applying the previous theorem

□

**Example 5.** *Go back to the example...*

Now we do the computation.