HOMEWORK 2

STA 624.01, Applied Stochastic Processes
Spring Semester, 2015

Due: Thursday, February 5, 2015

Readings: Chapter 1 of text.

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 In a simple model of the weather, each day is classified as 'sunny' or 'cloudy' and the sequence of weathers is modelled as a Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

Here the first row and column correspond to state 'sunny'.

(a) What is the probability that a cloudy day is followed by sunny one?

(b) What is the probability that a cloudy day is followed by two sunny ones in a row?

(c) If Friday is sunny, what is the probability that the following Sunday is sunny too?

(d) Compute P^2, P^4, P^8, P^{16} and P^{32} by repeatedly squaring the matrices you compute. What is your conclusion? What is the practical implication regarding the weather?

2 Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix:

$$P = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}.$$

a) What is the probability in the long run that the chain is in the state 1? Solve this problem by raising the matrix to a high power.

b) What is the probability in the long run that the chain is in the state 1? Solve this problem by directly computing the invariant probability.

3 A simple model of DNA is that the sequence of the bases A, C, G and T forms a Markov chain. Assume this chain has transition probability matrix

$$P = \begin{pmatrix} 0.3 & 0.22 & 0.21 & 0.27 \\ 0.28 & 0.22 & 0.30 & 0.20 \\ 0.23 & 0.32 & 0.23 & 0.22 \\ 0.18 & 0.22 & 0.30 & 0.30 \end{pmatrix}.$$

here the first row and column corresponds to A, etc. A subsequence that has been analysed reads ATGxxCGT, where 'xx' means that one is uncertain about these two bases. Biological considerations however suggest that these two symbols are either AC ot TG. Which of these two alternatives is the most probable?

4 Consider two urns A and B containing N balls. An experiment is performed in which one of the N balls is selected with probability depending on the urn contents (i.e., if A currently has k balls, a ball is chosen from A with probability k/N or from B with probability $(N - k)/N$). Then, an urn is selected and then depositing the selected ball in the selected urn. Urn A is chosen with probability k/N or urn B is chosen with probability $(N - k)/N$. Determine the transition matrix of the Markov chain with states represented by the contents of A.

5 A matrix $P = \{P_{i,j}\}$ is called *stochastic* if $P_{i,j} \geq 0$ for all i, j and $\sum_j P_{i,j} = 1$ for all i. A matrix P is called *doubly stochastic* if in addition to the above, $\sum_i P_{i,j} = 1$ for all j. Prove that if a finite irreducible Markov chain has doubly stochastic transition probability matrix, the stationary probabilities π_i for all i exist and are equal.

Computer Problems

For this problem, please print out all code used and all results. We will use the same Markov chain from the computer problem in HW1!

This Markov chain is called simple random walk with reflecting boundaries. The state space is $\{1, 2, \ldots, n\}$. It is defined as follows:

$$\begin{aligned}
P(X_{t+1} = i+1 | X_t = i) &= p, \ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = i-1 | X_t = i) &= 1-p, \ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = n-1 | X_t = n) &= 1-p \\
P(X_{t+1} = n | X_t = n) &= p \\
P(X_{t+1} = 1 | X_t = 1) &= 1-p \\
P(X_{t+1} = 2 | X_t = 1) &= p.
\end{aligned}$$

We will use $n = 10$ and $n = 15$ for $p = 0.2,\ 0.5,\ 0.6$. Graphically compare the quantities

$$t^{-1} \sum_{k=1}^{t} \mathbf{1}_{X_k = x},\ x \in \{1, \ldots n\},$$

where $\mathbf{1}$ is the indicator function and the histogram of the limiting distribution π for $t = 10^3,\ 10^4,\ 10^5$ if we start $X_0 = 1$.