HOMEWORK 3

STA 624.01, Applied Stochastic Processes
Spring Semester, 2015

Due: Tuesday, February 17, 2014

Readings: Chapter 2 of text,

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 Suppose P is a $s \times s$ stochastic matrix (i.e., the row sums equal to one) and

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{pmatrix}.$$

Show that P^2 is a $s \times s$ stochastic matrix and P^n is also a $s \times s$ stochastic matrix for all positive integer n.

2 Three different Markov chains are defined by the following transition matrices:

1.
$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

2.
$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.
$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

(a) Draw a transition graph for each chain. Is the MC irreducible?

(b) Identify the communication classes and classify them as periodic or aperiodic (if it is a periodic state the periodicity of it), transient or recurrent.

3 Assume this chain has transition probability matrix

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

(a) Show that the chain is irreducible, recurrent, and periodic. What is the periodicity of the chain?

(b) Find the unique stationary probability distribution.

4 A sequence of electrical impulses passes a measurement instrument that stores the largest value measured so far. Assume that the impulses at time points $0, 1, 2, 3, \cdots$ can be modelled as independent random variables $Y_0, Y_1, Y_2, Y_3, \cdots$ with a uniform distribution on $\{1, 2, 3, 4, 5\}$. Thus, if $X_1, X_2, X_3, \cdots$ are the values stored at time points $0, 1, 2, 3, \cdots$, then

$$X_n = \max(Y_0, Y_1, Y_2, \cdots, Y_n) \text{ for } n = 0, 1, 2, 3, \cdots .$$

Motivate that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain and write down the transition probability matrix.

5 Prove that if the Markov chain $\{X_n\}$ is irreducible then all states have the same periodecity.

Computer Problems

For this problem, please print out all code used and all results.

Consider the Markov chain I demonstrated in classes. We have two strings. We have two moves; one is twist and one is rotate. These moves correspond to the following maps:

$$\begin{array}{rl} x \to x+1 & \text{for twist} \\ x \to -\frac{1}{x} & \text{for rotate.} \end{array}$$

When two strings are untangled and aligned nicely we have $x = 0$ (this is always true!!!).

Let X_n be the number x after n moves and we start $X_0 = 0$.

The state space is the set of all rational numbers $\mathbb{Q}$. Each move is defined as follows:

$$\begin{aligned} P(X_{n+1} = x+1 | X_n = x) &= p, \quad \text{if } x \neq 0 \\ P(X_{n+1} = -1/x | X_n = x) &= 1-p, \quad \text{if } x \neq 0 \\ P(X_{n+1} = 1 | X_n = 0) &= 1 \end{aligned}$$

a) Write code for simulating this Markov chain.

b) Find the limiting distribution for $p = 0.2,\ 0.5,\ 0.6$ by simulating the chain multiple times starting from $X_0 = 0$.

c) With $p = 0.2,\ 0.5,\ 0.6$, estimate the expected number of steps needed to return to 0 starting at 0.