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A review of countable time MC

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Some definitions on MC

Definition A *discrete time stochastic process* is a collection of random variables $\{X_0, X_1, X_2, \dots\}$ defined on a common sample space and state space.

Definition A (time homogeneous) *Markov chain* is a stochastic process such that for all $n \in \{0, 1, \dots\}$ and $x_0, x_1, \dots, x_n, y \in S$,

$$P(X_{n+1} = y | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) = P(X_{n+1} = y | X_n = x_n).$$

[All Markov chains in this course will be time homogeneous unless explicitly stated otherwise.]

Finite State MC

Definition Given a Markov chain with a finite state space S , a *transition matrix* is a matrix whose entry in the i th row and j th column is

$$P(X_{n+1} = j | X_n = i).$$

Definition Given a Markov chain with finite state space S , the *transition graph* has vertex set S , and has directed edges (i, j) with weight

$$P(X_{n+1} = j | X_n = i)$$

whenever this weight is positive.

Definition π is a *stationary* or *invariant* distribution for a Markov chain if $\sum_{y \in S} \pi_y p(y, x) = \pi_x$.

Definition For countable state Markov chains, if

$$\sum_{x \in S} \pi_x P(X_{t+1} = y | X_t = x) = \pi_y$$

then π is a stationary distribution. These are called the *balance equations*.

Definition π is a *limiting* distribution for a countable state Markov chain if

$$\lim_{n \rightarrow \infty} P(X_n = i | X_0 = j) = \pi_i,$$

for all states i and j .

Definition States x and y of a Markov chain communicate if for some n and some m , $P(X_n = y|X_0 = x) > 0$ and $P(X_m = x|X_0 = y) > 0$.

Definition A Markov chain is *irreducible* if all states communicate. Otherwise it is reducible.

Definition The *period* of an irreducible Markov chain is

$$\gcd\{n : P(X_n = x|X_0 = x) > 0\}$$

for any state x . A chain with period 1 is called *aperiodic*.

Ergodic Theorem for finite state Markov chains

For irreducible aperiodic finite state Markov chains, a unique stationary distribution exists with $\pi_x > 0$ for all $x \in S$, and this will also be the limiting distribution. In addition, the expected time of return from x to itself will be $1/\pi_x$.

Countable State MC

Definition Let

$$R_x = \inf\{t > 0 : X_t = x | X_0 = x\}$$

be the *return time* for the state x . Then state x is *recurrent* if $P(R_x < \infty) = 1$. If any state in an irreducible chain is recurrent, they all are, and it is a *recurrent chain*.

Definition A state that is not recurrent is *transient*. An irreducible chain which is not recurrent is a *transient chain*.

Fact A state x is transient iff $E(N(x) | X_0 = x) < \infty$ where $N(x) = \sum_{m=1}^{\infty} 1_{\{X_m=x\}}$.

Definition Consider a chain that is recurrent and aperiodic. If there exist states x and y in S such that $\lim_{n \rightarrow \infty} P(X_n = y | X_0 = x) = 0$, the chain is *null recurrent*. When the limit is positive, it is *positive recurrent*.

Fact Let R be the time needed to return to state x given $X_0 = x$. For null recurrent chains, $E(R) = \infty$ and $P(R < \infty) = 1$.

Ergodic Thm for countable state space

For an irreducible aperiodic positive recurrent chain on a countable state space, there exists a unique stationary distribution π with $\pi_i > 0$. Also, π is the limiting distribution, and if R is the time for return to x starting from x , $E(R) = 1/\pi_x$. If the chain is null recurrent or transient, there is no stationary distribution π .

Definition A *branching process* is a special Markov chain on $\{0, 1, 2, \dots\}$ such that if ξ_1^n, ξ_2^n, \dots are i.i.d. draws from a distribution on the nonnegative integers with a mean μ , then

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^n.$$

Theorem: If $\mu \leq 1$ then the extinction probability equal to 1. If $\mu > 1$ then the extinction probability is less than 1 and equal to the smallest positive root of $a = \sum_{i=0}^{\infty} P(\xi = i)a^i$ with $0 < a < 1$.