HOMEWORK 7

STA 624.01, Applied Stochastic Processes
Spring Semester, 2016

Due: Tues April 5th, 2016

Readings: Chapter 3 of text

Regular Problems

1 (Lawler 3.5) Let X_t be a Markov chain with state space $\{1, 2\}$ and rates $q_{1,2} = 1$, $q_{2,1} = 4$. Let

$$P_t(i, j) = P(Y_t = j|Y_0 = i).$$

Find the matrix P_t.

2 (Lawler 3.7) Let X_t be a Markov chain with state space $\{1, 2, 3\}$ and rates $q_{1,2} = 1$, $q_{2,1} = 4$, $q_{2,3} = 1$, $q_{3,2} = 4$, $q_{1,3} = 0$, $q_{3,1} = 0$. Let

$$P_t(i, j) = P(Y_t = j|Y_0 = i).$$

Find the matrix P_t.

3 (Lawler 3.8) Consider the continuous-time Markov chain with the state space $\{1, 2, 3, 4\}$ and infinitesmal generator

$$Q = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find the stationary distribution $\bar{\pi}$.
(b) Suppose that the chain starts in state 1. What is the expected amount of time until it changes state for the first time?
(c) Again suppose that the chain starts in state 1. What is the expected amount of time until it changes to state 4?

4 (Lawler 3.10) Consider the continuous-time Markov chain with the state space $\{1, 2, 3, 4\}$ and infinitesmal generator

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find the stationary distribution $\bar{\pi}$.
(b) Suppose that the chain starts in state 1. What is the expected amount of time until it changes state for the first time?
(c) Again suppose that the chain starts in state 1. What is the expected amount of time until it changes to state 4?

5 **Bernoulli-Laplace model of diffusion**: Long before the invention of the term Markov chain, discrete time Markov chains were being studied, they just weren't called that. In the BL model of diffusion, two urns each contain m balls (so $2m$ balls total). A certain number of the balls, b are blue, and the rest $(2m - b)$ are green. Say $b \leq m$.

At each time step, pick one ball uniformly from each urn, and interchange them. Let X_t be the number of blue balls in the first urn.

(a) Find the transition probabilities for this Markov chain.
(b) Find the stationary distribution for this Markov chain.

Computer Problem: Branching Processes Consider a Lotka-Volterra predator-prey model where

$$\begin{aligned} X_t &= \text{number of prey} \\ Y_t &= \text{number of predators} \end{aligned}$$

with moves:

move	rate
$(1,0)$	aX_t
$(-1,0)$	bX_tY_t
$(0,1)$	cX_tY_t
$(0,-1)$	dY_t

Using the following parameters: $a = 1$, $b = .02$, $c = .01$, $d = 1$, $X_0 = 120$, $Y_0 = 40$, run the model forward five times. For each run plot two things. First, plot X_t and Y_t versus time up to about $t = 20$. Second, plot X_t versus Y_t over the same time frame.

Do the same two plots for a single run of length $t = 200$. For all your plots, you may abort your simulation if the predators go extinct. Just plot up to the time of extinction if that happens.