A review of countable time MC

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Ruriko Yoshida
Dept. of Statistics University of Kentucky

Some definitions on MC

Definition A *discrete time stochastic process* is a collection of random variables $\{X_0, X_1, X_2, \ldots\}$ defined on a common sample space and state space.

Definition A (time homogeneous) *Markov chain* is a stochastic process such that for all $n \in \{0, 1, \ldots\}$ and $x_0, x_1, \cdots, x_n, y \in S$,

$$P(X_{n+1} = y | X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = y | X_n = x_n).$$

[All Markov chains in this course will be time homogeneous unless explicitly stated otherwise.]

Finite State MC

Definition Given a Markov chain with a finite state space S, a *transition matrix* is a matrix whose entry in the ith row and jth column is

$$P(X_{n+1} = j | X_n = i).$$

Definition Given a Markov chain with finite state space S, the *transition graph* has vertex set S, and has directed edges (i, j) with weight

$$P(X_{n+1} = j | X_n = i)$$

whenever this weight is positive.

Definition π is a *stationary* or *invariant* distribution for a Markov chain if $\sum_{y \in S} \pi_y p(y, x) = \pi_x$.

Definition For countable state Markov chains, if

$$\sum_{x \in S} \pi_x P(X_{t+1} = y | X_t = x) = \pi_y$$

then π is a stationary distribution. These are called the *balance equations.*

Definition π is a *limiting* distribution for a countable state Markov chain if

$$\lim_{n \to \infty} P(X_n = i | X_0 = j) = \pi_i,$$

for all states i and j.

Definition States x and y of a Markov chain communicate if for some n and some m, $P(X_n = y|X_0 = x) > 0$ and $P(X_m = x|X_0 = y) > 0$.

Definition A Markov chain is *irreducible* if all states communicate. Otherwise it is reducible.

Definition The *period* of an irreducible Markov chain is

$$\text{gcd}\{n : P(X_n = x|X_0 = x) > 0\}$$

for any state x. A chain with period 1 is called *aperiodic*.

Ergodic Theorem for finite state Markov chains

For irreducible aperiodic finite state Markov chains, a unique stationary distribution exists with $\pi_x > 0$ for all $x \in S$, and this will also be the limiting distribution. In addition, the expected time of return from x to itself will be $1/\pi_x$.

Countable State MC

Definition Let

$$R_x = \inf\{t > 0 : X_t = x | X_0 = x\}$$

be the *return time* for the state x. Then state x is *recurrent* if $P(R_x < \infty) = 1$. If any state in an irreducible chain is recurrent, they all are, and it is a *recurrent chain.*

Definition A state that is not recurrent is *transient*. An irreducible chain which is not recurrent is a *transient chain*.

Fact A state x is transient iff $E(N(x)|X_0 = x) < \infty$ where $N(x) = \sum_{m=1}^{\infty} 1_{\{X_m = x\}}$.

Definition Consider a chain that is recurrent and aperiodic. If there exist states x and y in S such that $\lim_{n\to\infty} P(X_n = y|X_0 = x) = 0$, the chain is *null recurrent*. When the limit is positive, it is *positive recurrent*.

Fact Let R be the time needed to return to state x given $X_0 = x$. For null recurrent chains, $E(R) = \infty$ and $P(R < \infty) = 1$.

Ergodic Thm for countable state space

For an irreducible aperiodic positive recurrent chain on a countable state space, there exists a unique stationary distribution π with $\pi_i > 0$. Also, π is the limiting distribution, and if R is the time for return to x starting from x, $E(R) = 1/\pi_x$. If the chain is null recurrent or transient, there is no stationary distribution π.

Definition A *branching process* is a special Markov chain on $\{0, 1, 2, \ldots\}$ such that if $\xi_1^n, \xi_2^n, \ldots$ are i.i.d. draws from a distribution on the nonnegative integers with a mean μ, then

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^n.$$

Theorem: If $\mu \leq 1$ then the extinction probability equal to 1. If $\mu > 1$ then the extinction probability is less than 1 and equal to the smallest positive root of $a = \sum_{i=0}^{\infty} P(\xi = i) a^i$ with $0 < a < 1$.