HOMEWORK 3

STA701.01, Statistical Inference
Fall Semester, 2013

Due: Thurs, Oct 10th 2013

1 Two contestants simultaneously put up either one or two fingers. One of the players, call her player I, wins if the sum of the digits showing is odd, and the other player, player II, wins if the sum of the digits showing is even. The winner in all cases receives, from the loser, in dollars the sum of the digits showing.

(a) Define $\mathcal{S}, \mathcal{A}$ and the loss function $\ell(\theta, a)$ for this game.
(b) Find the minimax action.
(c) Plot the set of randomized actions for this game.
(d) Find the minimax randomized action.
(e) For which prior is the randomized minimax action bayes?

2 Let $\mathcal{S} = \{1/3, 2/3\}$, let $\mathcal{A}$ be the whole real line and the loss function $\ell(\theta, a) = (\theta - a)^2$. A coin is tossed with probability of heads θ (one of the two states of nature). The data space is $\{H, T\}$ for heads and tails.

(a) Define the set of nonrandomized decision rules D.
(b) Show that points in D can be naturally plot as points on the Euclidean xy plane. Show the plot.
(c) Find the Bayes rule with respect to an arbitrary prior.
(d) Show the set of Bayes rules as a subset of the plane.
(e) Find the minimax decision rule.

3 Let $\mathcal{S} = \mathcal{A} = \{0, 1\}$ and the loss function be zero if the action matches the state of nature and one if it doesn't. In other words the statistician (i.e. player) tries to guess what nature has chosen. The statistician observes the value of X with probability distribution:

$$P\{X = x|\theta\} = 2^{-k} \quad \text{if } x = k - \theta \text{ for } k = 1, 2, \ldots$$

Describe the set of all nonrandomized decision rules. Plot the risk set S in the plane. i.e., label a decision rule with its vector of risks on the xy plane.

(a) Find the minimax decision rule.
(b) Find the least favorable prior distribution for the statistician.
(c) Show that the bayes rule w.r.t. the least favorable distribution is the minimax rule.

4 A natural loss function for estimating the probability θ of a coin is the relative entropy given by,

$$\ell(\theta, a) = \theta \log \frac{\theta}{a} + (1 - \theta) \log \frac{1 - \theta}{1 - a}$$

for a, θ both in $[0, 1]$.

(a) Find the minimax estimator of θ for this loss function.
(b) Find the bayes risk for the uniform prior.
(c) Find the bayes action for the uniform prior.

5 A coin with unknown probability of heads θ is tossed n times producing x heads. We want to compare the performance of the observed frequency of heads x/n (a.k.a. the Maximum Likelihood Estimator or MLE) with the Bayes estimator (bayes rule) w.r.t. the beta prior with parameters $\alpha > 0$ and $\beta > 0$ and quadratic loss.

(a) Reparametrize the beta prior in terms of the prior mean μ and prior strength (or equivalent number of prior observations) ν where,

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and } \nu = \alpha+\beta$$

and show that the bayes rule $\delta_B(x)$ is a weighted average between the prior mean μ and observed mean x/n with weights ν and n respectively.

(b) Find the risks $R(\theta, \delta_B)$ and $R(\theta, \delta_0)$ where, $\delta_0(x) = x/n$.

(c) Show that the Bayes rule wins (in terms of risk) in the region,

$$\frac{(\mu-\theta)^2}{\theta(1-\theta)} \leq \frac{2}{\nu} + \frac{1}{n}$$

6 That pesky unknown probability of heads θ again!. But now the loss is not quadratic but,

$$\ell(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$$

just like the left hand side of the previous example!. As before we have $X \sim \text{Bin}(n,\theta)$ and $\theta \sim \text{Beta}(\alpha,\beta)$.

(a) Find the bayes estimator of θ and be careful about the treatment of $x = 0$ and $x = n$.

(b) Find the minimax estimator of θ.

(c) Explain the tittle for this problem.