HOMEWORK 0

STA703.01, Advanced Probability
Fall Semester, 2015

Due: Thurs Sept 3rd, 2015

1 [Rudin, Theorem 7.5]
Suppose E is a set and suppose $f, f_1, f_2, \ldots$ are functions from E to $\mathbb{R}$.
Let

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Let

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then show $f_n \to f$ uniformly if and only if $M_n \to 0$ as $n \to \infty$.

2 For each f_n and E, find the limit f and show whether $f_n \to f$ uniformly or not.
(a) $f_n = x^n$ on $E = (0, 1)$.
(b) $f_n = x^n$ on $E = (0, \alpha)$, $0 < \alpha < 1$.
(c) $f_n = \frac{1}{n\cdot x+1}$ on $E = (0, 1)$.
(d) $f_n = \frac{1}{n\cdot x+1}$ on $E = (\alpha, \infty)$, where $\alpha > 0$.
(e) $f_n = \frac{x}{n\cdot x+1}$ on $E = [0, \infty)$.
(f) $f_n = x^n(1-x)$ on $E = [0, 1]$.

3 Prove that if $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E, then $\{f_n(x)\}$ converges uniformly to 0.

4 For each $\sum_{n=1}^{\infty} f_n$ and E, show whether $\sum_{n=1}^{\infty} f_n$ converges uniformly or not.
(a) $\sum_{n=1}^{\infty} \frac{n\cdot x}{1+n^5 x^2}$ on $E = \mathbb{R}$.
(b) $\sum_{n=1}^{\infty} e^{-n\cdot x^2}$ on $E = (0, 1)$.
(c) $\sum_{n=1}^{\infty} e^{-n\cdot x^2}$ on $E = (\alpha, \infty)$, where $\alpha > 0$.
(d) $\sum_{n=1}^{\infty} x \cdot e^{-n\cdot x}$ on $E = [0, 1]$.
(e) $\sum_{n=1}^{\infty} x \cdot e^{-n^2\cdot x}$ on $E = [0, \infty)$.

5 Let f be differentiable on $E \subset \mathbb{R}$. Suppose there exists M such that $|f'(x)| \le M, \forall x \in E$. Then show that f is uniformly continuous on E.

6 For each f and E, show whether f is uniformly continuous on E.
(a) $f(x) = x\cos\frac{1}{x}$ on $E = (0, a)$, $a > 0$.
(b) $f = x\cos\frac{1}{x}$ on $E = (a, \infty)$, $a > 0$.
(c) $f(x) = \sin\frac{1}{x}$ on $E = (0, a)$, $a > 0$.
(d) $f(x) = \sin\frac{1}{x}$ on $E = (a, \infty)$, $a > 0$.
(e) $f(x) = \sqrt{x}$ on $E = [0, a)$, $a > 0$.
(f) $f(x) = \sqrt{x}$ on $E = [a, \infty)$, $a > 0$.