Practice MT exam, Probability

STA703.01, Advanced Probability

1 Random variable X and Y_n, $n = 1, 2, \ldots$ are defined on the same sample space with probability mass functions:

$$f_X(x) = \begin{cases} 1/2 & x = -1 \\ 1/2 & x = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_{Y_n}(y) = \begin{cases} 1/2 - 1/(n+1) & y = -1 \\ 1/2 + 1/(n+1) & y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For each of the following statement, prove that it is true or provide a counter example.

(a) $Y_n \to_d X$.

(b) $Y_n \to_p X$.

(c) $Y_n \to X$ a.s.

(d) Does the convergence of Y_n to X in distribution (in prob) (almost sure) still valid if they are independent?

2 (a) State the L_2 norm weak law of large numbers.

(b) Prove it.

(c) Show an example we can apply the L_2 norm weak law of large numbers.

3 Find the Lebesgue integral of $f(x) = xcos(x)$ over the interval $[-1, 1]$ with μ is Lebeasgue measure.

4 Let φ and ϕ be a simple function on a measure space $(\Omega, \mathcal{F}, \mu)$ with partitions

$$\varphi = \sum_{i=1}^{n} \alpha_i \mathbb{I}_{A_i},$$

$$\phi = \sum_{j=1}^{m} \beta_j \mathbb{I}_{B_j}.$$

(a) Show that

$$\varphi + \phi = \sum_{i}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \mathbb{I}_{A_i \cap B_j}.$$

(b) Also show that

$$\int \varphi + \phi d\mu = \int \varphi d\mu + \int \phi d\mu.$$

5 Prove if true, or give a counterexample if false.

(a) If $X_n \to X$ in probability, then $X_n \to X$ in distribution.

(b) Let c be a constant (or, if you prefer, a random variable placing all of it probability on c). If $X_n \to c$ in distribution, then $X \to c$ in probability.

(c) If for every $\epsilon > 0$, $\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty$, then $X_n \to 0$ almost surely.

6 Define X_1 and X_2 as independent if and only if $P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1)P(X_2 \in B_2)$ for all Borel sets B_1 and B_2.

(a) Suppose a random variable Y is independent of itself. Prove that there exists a real number c such that $P(Y = c) = 1$.

(b) Suppose that $g_1 : \Re \to \Re$ and $g_2 : \Re \to \Re$ are Borel measurable functions. If X_1 and X_2 are independent random variables, prove that $g_1(X_1)$ and $g_2(X_2)$ are independent.

(c) Suppose that X is a random variable and $g : \Re \to \Re$ is Borel measurable. Further, suppose that X and $g(X)$ are independent. What can be said in this case? (Prove your claim)

7 Let P be a probability measure on $(\Re, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra of Borel sets. Let F be the distribution function of P. That is, for $x \in \Re$, $F(x) = P((-\infty, x])$.

(a) Prove that $F(x)$ is right continuous.

(b) Prove that $F(x)$ has only countably many discontinuities. (You may assume that $\lim_{y \to x^-} F(y)$ exists and $\lim_{y \to x^-} F(y) \le \lim_{y \to x^+} F(y)$ for all x.)