Homework 8 Solutions

4.21 Let X = hour of operation in which the defective board was produced

Since the number of defectives is Poisson, then given that one defective was produced, actual time of occurance is equally likely in any small subinterval of time of a given size, and thus $X \sim U(0,8)$.

(a) $P(X<1) = \int_0^1 \frac{1}{8}\,dx = \frac{1}{8}x\Big|_0^1 = \frac{1}{8}$

(b) $P(X>7) = \int_7^8 \frac{1}{8}\,dx = \frac{1}{8}x\Big|_7^8 = \frac{1}{8}(8) - \frac{1}{8}(7) = 1 - \frac{7}{8} = \frac{1}{8}$

(c) $P(4<X\le 5 \mid X>4) = \dfrac{P(4<X\le 5, X>4)}{P(X>4)} = \dfrac{P(4<X\le 5)}{P(X>4)} = \dfrac{\int_4^5 \frac{1}{8}\,dx}{\int_4^8 \frac{1}{8}\,dx} = \frac{1}{4}$

4.35

(a) Note that since $\int_0^\infty x^{n-1} e^{-x/\theta}\,dx = \Gamma(n)\theta^n$, then for k integer valued we have $E[X^k] = \frac{1}{\theta}\int_0^\infty x^k e^{-x/\theta}\,dx = \frac{1}{\theta}\Gamma(k+1)\theta^{k+1} = \theta^k k!$

So $E[X] = (10)1! = 10$

$E[X^2] = 10^2\, 2! = 200$

$E[X^3] = 10^3\, 3! = 6000$

$E[X^4] = 10^4\, 4! = 240{,}000$

and $E[C] = 100 + 40E[X] + 3E[X^2] = 100 + 40(10) + 3(200) = 1100$

$E[C^2] = E\{[100 + 40(X) + 3(X^2)]^2\}$

$= 10{,}000 + 8{,}000\,E[X] + 2{,}200\,E[X^2] + 240\,E[X^3] + 9E[X^4]$

$= 10{,}000 + 8{,}000(10) + 2{,}200(200) + 240(6000) + 9(240{,}000)$

$= 4{,}130{,}000$

$V(C) = E[C^2] - (E[C])^2 = 4{,}130{,}000 - (1{,}100)^2 = 2{,}920{,}000$

(a) $E[L] = 30E[Y] + 2E[Y^2] = 30\alpha\beta + \dfrac{2\beta^2\Gamma(\alpha+2)}{\Gamma(\alpha)} = 30\alpha\beta + \dfrac{2\beta^2(\alpha+1)!}{(\alpha-1)!}$

$= 30(3)(2) + \dfrac{2(2)^2(24)}{2} = 276$

$V(L) = E[L^2] - (E[L])^2 = E[(30Y+2Y^2)^2] - (276)^2$

$= 900E[Y^2] + 120E[Y^3] + 4E[Y^4] - (276)^2$

$= 900(2)^2\dfrac{\Gamma(5)}{\Gamma(3)} + 120(2)^3\dfrac{\Gamma(6)}{\Gamma(3)} + 4(2)^4\dfrac{\Gamma(7)}{\Gamma(3)} - (276)^2$

$= 900(2)^2\dfrac{24}{2} + 120(2)^3\dfrac{120}{2} + 4(2)^4\dfrac{720}{2} - (276)^2$

$= 47{,}664$

(b) By Chebychev, we want K such that $1 - \frac{1}{K^2} = .89$. So $K=3$.

Then $[E[L] - 3\sqrt{V(L)}, E[L] + 3\sqrt{V(L)}]$

$= (276 - 3\sqrt{47{,}644}, 276 + 3\sqrt{47{,}644})$

$= (-378.963, 930.963)$

Since L is nonnegative, the interval is $(0, 930.963)$.

4.50 (a) Let $Y = X_1 + X_2$. Then $Y \sim \text{Gamma}(\alpha = 50+20 = 70, \beta = 2)$

then $E[Y] = \alpha\beta = 70 \cdot 2 = 140$

$V(Y) = \alpha\beta^2 = 70 \cdot 2^2 = 280$

$$f_Y(y) = \begin{cases} 0, & y \le 0 \\ \dfrac{1}{2^{70}\Gamma(70)} y^{69} e^{-y/2} = \dfrac{1}{2^{70}\,69!} y^{69} e^{-y/2}, & y > 0 \end{cases}$$

(b) Using Chebychev with $K=4$, we have

$P(Y>c) = P(Y-140 > c-140) \le P\left(|Y-140| > \dfrac{c-140}{\sqrt{280}}\sqrt{280}\right) < \dfrac{1}{16}$

for $\dfrac{c-140}{\sqrt{280}} = K = 4$.

Then $c = 4\sqrt{280} + 140 = 206.93$

(b) $P(C>2{,}000) = P(3X^2 + 40X + 100 > 2{,}000)$

$= P(3X^2 + 40X - 1{,}900 > 0)$

$= P[(X-r_1)(X-r_2) > 0]$ where $r_1 = \frac{10}{3}(-2+\sqrt{61}) = 19.36\,75$

and $r_2 = \frac{10}{3}(-2-\sqrt{61}) = -32.7$

Then $P(C>2{,}000) = P(X-r_1>0, X-r_2>0) +$

$P(X-r_1<0, X-r_2<0)$

$= P(X>r_1, X>r_2) + P(X<r_1, X<r_2)$

$= P(X>r_1) + P(X<r_2)$

$= P(X>r_1)$

$= 1 - (1 - e^{-r_1/10})$

$= e^{-1.93675}$

$= 0.1442$

4.37 Let X = tire life length

Then $X \sim Exp(30)$.

(a) $P(X>30) = 1 - F(30) = 1 - (1 - e^{-30/30}) = e^{-1} = 0.3679$

(b) $P(X>30 \mid X>15) = P(X > 30-15) = P(X>15) = 1 - F(15)$

$= 1 - (1 - e^{-15/30}) = e^{-1/2} = 0.6065$

4.47 For k integer valued,

$$E[Y^k] = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha+k-1} e^{-y/\beta} dy$$

$$= \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\Gamma(\alpha+k)\beta^{\alpha+k}} y^{\alpha+k-1} e^{-y/\beta} dy$$

$$= \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}$$