Name Solution

STA 524 Midterm 2

Probability

November 12th, 2007

There are five questions on this test. DO use calculators if you need them. "Do that one thing, and then that other thing" is not a valid answer. There will be no bathroom break allowed. Divination to obtain answers is not allowed.

Obligatory Hitchhikers Guide to the Galaxy Notice: Don't Panic!!!

You have 50 minutes to complete this test. Please ask me questions if a question needs clarification.

Each question is worth the same number of points.

Question 1: Proof

(a) Define the variance of a r.v X.

$Var(X) = \mathbb{E}[(X-\mu)^2]$ where $\mu = \mathbb{E}(X) < \infty$

& if $\mathbb{E}(X)$ exists.

(b) Using (a), show

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

$Var(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$

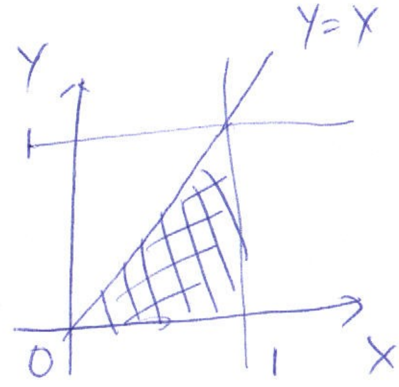
$= \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2$

$= \mathbb{E}(X^2) - 2\mu^2 + \mu^2$

$= \mathbb{E}(X^2) - \mu^2$

$= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$ ✓

Question 2: Joint PDF



Suppose r.v. X and Y have the following pdf:

$$f(x,y) = \begin{cases} 2(x+y) & \text{for } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that the condition for $f(x,y)$ is different from the book).

(a) Compute the marginal pdf of X.

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x (2x+2y)\,dy$$

$$= 2xy + y^2 \Big|_0^x = 2x^2 + x^2 = \boxed{3x^2}$$

(b) Compute $Pr(X < 1/2)$.

$$Pr(X < 1/2) = \int_{-\infty}^{1/2} f_1(x)\,dx = \int_0^{1/2} 3x^2\,dx = x^3 \Big|_0^{1/2} = \boxed{1/8}$$

(c) Compute the conditional pdf of Y given $X = x$.

$$g_2(y|x) = \frac{f(x,y)}{f_1(x)} = \boxed{\frac{2x+2y}{3x^2}} \quad \text{for } 0<y<x<1$$

Question 3: Moments
Suppose a r.v. X has the following mgf:

$$\psi(t) = \frac{1}{4}(3e^t + e^{-t}),\ t \in \mathbb{R}.$$

(a) Compute the expected value of X.

$$\mathbb{E}(X) = \psi'(0) = \frac{3}{4}e^t - \frac{1}{4}e^{-t}\Big|_{t=0} = \frac{3}{4} - \frac{1}{4} = \boxed{1/2}$$

(b) Compute the variance of X.

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathbb{E}(X^2) = \psi''(0) = \frac{3}{4}e^t + \frac{1}{4}e^{-t}\Big|_{t=0} = \frac{3}{4} + \frac{1}{4} = 1$$

$$Var(X) = 1 - \left(\frac{1}{2}\right)^2 = \boxed{\frac{3}{4}}$$

Question 4: Expectations and variances

Suppose that a r.v. X has the mean μ and the variance σ^2 and let $Y = aX + b$. Determine the value of a and b where $\mathbb{E}(Y) = 0$ and $Var(Y) = 1$.

$$\mathbb{E}(Y) = \mathbb{E}(aX+b) = a\mathbb{E}(X)+b = a\mu + b = 0$$

$$Var(Y) = Var(aX+b) = a^2 Var(X) = a^2\sigma^2 = 1$$

$$\Rightarrow a = \pm\frac{1}{\sigma}.$$

$$\boxed{\begin{array}{ll} \text{If } a = \frac{1}{\sigma} & b = \frac{-\mu}{\sigma}. \\ \text{If } a = \frac{-1}{\sigma}, & b = \frac{+\mu}{\sigma}. \end{array}}$$

Question 5: Covariance and correlations

Suppose r.v. X and Y have $\mathbb{E}(X) = 3$, $\mathbb{E}(Y) = 1$, $Var(X) = 4$, and $Var(Y) = 9$. Let $Z = 5X - Y$.

(a) Compute $\mathbb{E}(Z)$ and $Var(Z)$ if X and Y are independent.

$$\mathbb{E}(Z) = \mathbb{E}[5X - Y] = 5\mathbb{E}[X] - \mathbb{E}(Y) = 15 - 1 = \boxed{14}$$

$$Var(Z) = Var(5X - Y) = Var(5X) + Var(-Y)$$

$$= 25 Var(X) + (-1)^2 Var Y$$

$$= 100 + 9 = \boxed{109}$$

(b) Compute $\mathbb{E}(Z)$ and $Var(Z)$ if the covariance of X and Y is 0.25.

$$\mathbb{E}(Z) = 14.$$

$$Var(Z) = 5^2 Var(X) + (-1)^2 Var(Y) + 2(5)(-1) cov(X,Y)$$

$$= 25 \cdot 4 + 9 - 10 \cdot (0.25)$$

$$= 109 - 2.5 = \boxed{106.5}$$

(c) Compute $\mathbb{E}(Z)$ and $Var(Z)$ if the correlation of X and Y is 0.25.

$$\rho = \frac{cov(X,Y)}{\sigma_X \sigma_Y} \Rightarrow cov(X,Y) = \rho \cdot \sigma_X \cdot \sigma_Y = 0.25(2)\cdot(3)$$

$$= 1.5$$

$$\mathbb{E}(Z) = 14$$

$$Var(Z) = 25 \cdot 4 + 9 - 10 \cdot (1.5)$$

$$= 109 - 15 = \boxed{94}$$