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A review of Chapter 3 and Chapter 4

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Discrete random variables

Definition A **random variable** X is a function from Ω to \mathbb{R} .

Definition We say X a **discrete random variable** if X can take a sequence of different values.

Definition If X is a discrete random variable and has a discrete distribution the **probability function** of X is defined as the function f s.t.

$$f(x) = Pr(X = x), \forall x \in \mathbb{R}.$$

Continuous random variables

Definition We say X a **continuous random variable** if there is a continuous nonnegative function f defined on \mathbb{R} such that

$$\Pr(X \in A) = \int_A f(x)dx, \forall A \subset \mathbb{R}.$$

This function f is called the **probability density function** of X .

Note For every p.d.f f of X must satisfy:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

Distribution functions

Definition The **distribution function** F of a r.v. X is a function defined for each real number $x \in \mathbb{R}$ s.t.

$$F(x) = Pr(X \leq x), x \in \mathbb{R}.$$

Thm The function $F(x)$ is non-decreasing as x increases.

Thm

$$\lim_{x \rightarrow -\infty} F(x) = 0 \text{ and } \lim_{x \rightarrow \infty} F(x) = 1.$$

Thm

$$F(x) = F(x^+), \forall x \in \mathbb{R}.$$

Distribution functions

Thm

$$\Pr(X > x) = 1 - F(x).$$

Thm

$$\Pr(x_1 \leq X \leq x_2) = F(x_2) - F(x_1), \forall x_1 \leq x_2 \in \mathbb{R}.$$

Thm

$$\Pr(X < x) = F(x^-), \forall x \in \mathbb{R}.$$

Thm

$$\Pr(X = x) = F(x) - F(x^-), \forall x \in \mathbb{R}.$$

Joint probability (density) functions

Definition Suppose X and Y are discrete r.v. Then the **joint probability function** of X and Y is defined as a function f s.t.

$$f(x, y) = Pr(X = x \text{ and } Y = y), \forall x, y \in \mathbb{R}.$$

Definition Suppose X and Y are continuous r.v. If there is a nonnegative continuous function f defined over \mathbb{R}^2 s.t.

$$Pr[(X, Y) \in A] = \int \int_A f(x, y) dx dy, \forall A \subset \mathbb{R}^2,$$

then this function $f(x, y)$ is called the **joint probability density function** of X and Y .

Joint probability density functions

Suppose $f(x, y)$ is the joint density function of X and Y . Then it must satisfy:

1. $f(x, y) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

Joint distribution functions

Definition The **joint distribution function** of X and Y is defined as a function F over \mathbb{R}^2 s.t.

$$F(x, y) = Pr[X \leq x \text{ and } Y \leq y], \forall x, y, \in \mathbb{R}.$$

Note

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) du dv \text{ and } f(x, y) = \partial^2 F(x, y) / \partial x \partial y.$$

Marginal distributions

Definition If X and Y are discrete r.v., then the **marginal probability function** of X is

$$f_1(x) = Pr(X = x) = \sum_y Pr(X = x \text{ and } Y = y) = \sum_y f(x, y).$$

The **marginal probability function** of Y is

$$f_2(y) = Pr(Y = y) = \sum_x Pr(X = x \text{ and } Y = y) = \sum_x f(x, y).$$

Marginal distributions

Definition If X and Y are continuous r.v., then the **marginal probability density function** of X is

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

The **marginal probability density function** of Y is

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Marginal distributions

Definition If X and Y are discrete r.v., then the **marginal distribution function** of X is

$$F_1(x) = Pr(X \leq x) = \sum_y Pr(X \leq x \text{ and } Y \leq y) = \sum_y F(x, y).$$

The **marginal distribution function** of Y is

$$F_2(y) = Pr(Y \leq y) = \sum_x Pr(X \leq x \text{ and } Y \leq y) = \sum_x F(x, y).$$

Marginal distributions

Definition If X and Y are continuous r.v., then the **marginal distribution function** of X is

$$F_1(x) = Pr(X \leq x) = \int_{-\infty}^{\infty} Pr(X \leq x \text{ and } Y \leq y)dy = \int_{-\infty}^{\infty} F(x, y)dy.$$

The **marginal distribution function** of Y is

$$F_2(y) = Pr(Y \leq y) = \int_{-\infty}^{\infty} Pr(X \leq x \text{ and } Y \leq y)dx = \int_{-\infty}^{\infty} F(x, y)dx.$$

Independent

Definition We say r.v. X and Y are **independent** if

$$Pr(X \in A \text{ and } Y \in B) = Pr(X \in A)Pr(Y \in B), \forall A, B \subset \mathbb{R}.$$

Note X and Y are independent iff

$$Pr(X \leq x, Y \leq y) = Pr(X \leq x)Pr(Y \leq y) \text{ iff}$$

$$F(x, y) = F_1(x)F_2(y) \text{ iff}$$

$$f(x, y) = f_1(x)f_2(y).$$

Independent

Definition We say n r.v. X_1, \dots, X_n are **independent** iff $\forall A_1, \dots, A_n \subset \mathbb{R}$,

$$Pr(X_1 \in A_1, \dots, X_n \in A_n) = Pr(X_1 \in A_1) \cdots Pr(X_n \in A_n).$$

Note X_1, \dots, X_n are independent iff

$$\begin{aligned} f(x_1, \dots, x_n) &= f_1(x_1) \cdots f_n(x_n) \text{ iff} \\ F(x_1, \dots, x_n) &= F_1(x_1) \cdots F_n(x_n). \end{aligned}$$

Quantile functions

Definition The **quantile function** is defined by

$$F^{-1}(p) = \min\{x \in \mathbb{R} | p \leq F(x), p \in (0, 1)\}.$$

Note

1. $F^{-1}(p)$ is a quantile of order p .
2. If x is a quantile of order p , then $F^{-1}(p) \leq x$.

Quantile functions

Thm

1. $F^{-1}(p)$ is increasing on $(0, 1)$.
2. $F^{-1}(p)$ is continuous from the left and has limit from the right.
3. $F(F^{-1}(p)) \geq p, \forall p \in (0, 1)$.
4. $F^{-1}(F(x)) \leq x, \forall x \in \mathbb{R}$.

Expectations

Definition If a r.v. X has a discrete distribution,

$$\mathbb{E}(X) = \sum_{w \in \Omega} X(w) Pr(\{x\}) = \sum_x x f(x).$$

Definition If a r.v. X has a continuous distribution,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

Expectations

Note We say $\mathbb{E}(X)$ exists iff $\int_{-\infty}^{\infty} |x|f(x)dx < \infty$.

Thm For a nonnegative continuous r.v. $X \geq 0$,

$$\mathbb{E}(x) = \int_0^{\infty} (1 - F(x))dx.$$

Thm If $Y = aX + b$, where $a, b \in \mathbb{R}$ are constant, then

$$\mathbb{E}(Y) = a\mathbb{E}(X) + b.$$

Thm If there is a constant $a \in \mathbb{R}$ s.t. $Pr(X \geq a) = 1$, then $\mathbb{E}(X) \geq a$. If there is a constant $b \in \mathbb{R}$ s.t. $Pr(X \leq b) = 1$, then $\mathbb{E}(X) \leq b$.

Expectations

Thm If X_1, \dots, X_n are r.v. s.t. $\mathbb{E}(X_i)$ exists for all i ,

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n).$$

Thm If X_1, \dots, X_n are independent r.v. s.t. $\mathbb{E}(X_i)$ exists for all i ,

$$\mathbb{E}\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n \mathbb{E}(X_i).$$

Thm If X_1, \dots, X_n form a random sample with mean μ and if $Y = X_1 + \dots + X_n$ and $M = \frac{1}{n}X_1 + \dots + X_n$. Then

$$\mathbb{E}(Y) = n\mu \text{ and } \mathbb{E}(M) = \mu.$$

Variance

Definition For a r.v. X with a finite mean, the **variance** is defined to be

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2].$$

The **standard deviation** of X is $\sqrt{\text{Var}(X)}$.

Thm $\text{Var}(X) = 0$ iff there is a constant c s.t. $\text{Pr}(X = c) = 1$.

Thm For constants $a, b \in \mathbb{R}$, $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Thm

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Thm If X_1, \dots, X_n are independent r.v.,

$$\text{Var}(X_1 + \dots + X_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n).$$

Moments

Definition The expectation $\mathbb{E}(X^k)$ for a positive integer k is called the **k th moment** of X .

Note We say the k th moment exists iff $\mathbb{E}(|X|^k) < \infty$.

Thm If $\mathbb{E}(|X|^k) < \infty$ for some k then $\mathbb{E}(|X|^j) < \infty$ for all positive integer $j < k$.

Definition The expectation $\mathbb{E}[(X - \mu)^k]$ for a positive integer k is called the **k th central moment** of X .

Moment generating functions

Definition The **moment generating function** $\psi(t)$ is defined

$$\psi(t) = \mathbb{E}(\exp(tX)).$$

Note If there is a derivative ψ' around $t = 0$, then

$$\psi'(0) = \mathbb{E}(X).$$

If there is a derivative ψ^k around $t = 0$, then

$$\psi^k(0) = \mathbb{E}(X^k).$$

Moment generating functions

Thm Let X be a r.v. with the mgf ψ_1 and let $Y = aX + b$ where $a, b \in \mathbb{R}$ are constant and ψ_2 is the mgf of Y . Then

$$\psi_2(t) = \exp(bt)\psi_1(at).$$

Thm If X_1, \dots, X_n are independent r.v. with the mgf ψ_1, \dots, ψ_n , respectively, then

$$\psi(t) = \prod_{i=1}^n \psi_i(t), \forall t \text{ with } \psi_i(t) \text{ exit,}$$

where ψ is the mgf of $X_1 + \dots + X_n$.

Medians

Definition A **median** of the distribution of X is defined as a number m s.t.

$$\Pr(X \leq m) \geq 1/2 \text{ and } \Pr(X \geq m) \geq 1/2.$$

Definition The **mean absolute error** is defined as $\mathbb{E}(|X - d|)$.

Thm Let m be a median of the distribution of X and $d \in \mathbb{R}$. then

$$\mathbb{E}(|X - m|) \leq \mathbb{E}(|X - d|).$$

Covariance and correlations

Definition The **covariance** of X and Y is defined

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)].$$

Definition If $0 < \sigma_X^2 < \infty$ and $0 < \sigma_Y^2 < \infty$, then the **correlation** of X and Y is defined

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Covariance and correlations

Thm (Schwarz Inequality)

$$[\mathbb{E}(XY)]^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2).$$

Thm If $\sigma_X^2 < \infty$ and $\sigma_Y^2 < \infty$, then

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

Thm If X and Y are independent and $0 < \sigma_X^2 < \infty$ and $0 < \sigma_Y^2 < \infty$,

$$\text{Cov}(X, Y) = \rho(X, Y) = 0.$$

Covariance and correlations

Thm Let X be a r.v with $0 < \sigma_X^2 < \infty$ and let $Y = aX + b$ where $a, b, \in \mathbb{R}$ are constant. If $a > 0$, then $\rho(X, Y) = 1$ and if $a < 0$, then $\rho(X, Y) = -1$.

Thm If X and Y are r.v. s.t. $Var(X) < \infty$ and $Var(Y) < \infty$, then

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y).$$

Thm If X_1, \dots, X_n are r.v. with $Var(X_i) < \infty$, then

$$Var(X_1 + \dots + X_n) = Var(X_1) + \dots + Var(X_n) + 2 \sum_{i < j} Cov(X_i, X_j).$$