HOMEWORK 2

STA 624.01, Applied Stochastic Processes
Fall Semester, 2005

Due: Friday, January. 26, 2007

Readings: Chapter 1 of text, Tutorial 2 on MATLAB

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 Consider the telecommunications Markov chain with transition matrix:

$$A = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

Suppose I flip a fair coin to start the chain in the 0 or 1 position.
a) What is the probability vector for X_0?
b) What is the distribution of X_1 (the chain after one step)?
c) What is the distribution of X_2 (the chain after two steps)?

2 Consider a Markov chain on states $\{a, b, c\}$ with the following transition matrix:

$$A = \begin{pmatrix} .2 & .3 & .5 \\ .7 & .1 & .2 \\ .4 & .4 & .2 \end{pmatrix}$$

Compute (a) $P(X_{20} = a|X_0 = a)$ and (b) $P(X_{10} = c, X_{15} = b|X_0 = a)$.

3 Rudy's Mafia spends each day doing one of three things: extortion, gambling, breaking thumbs. Each day a random choice of what to do is based on the last day. When the previous day's activity was extortion or breaking thumbs, there is a 1/3 chance for each activity the next day. When the day's activity was gambling, there is an 80% chance of gambling the next day, a 10% chance of extorting, and a 10% chance of breaking thumbs.

a) After a large number of days, what is the probability that the day is spent gambling?

b) The first day is spent gambling. What is the probability that the fourth day is spent breaking thumbs and the sixth day is spent extorting?

4 Random walk around a circle. Consider the state space $\{0, 1, \ldots, 7\}$. Let $X_1, X_2, \ldots$ be successive rolls of a fair 6-sided die. Let

$$S_n = S_{n-1} + X_n - 8 \cdot 1(S_{n-1} + X_n \geq 8).$$

a) Write out the transition matrix for this transition function.
b) What is the limiting probability that the chain is in state 7?

Computer Problems

For this problem, please print out all code used and all results.

This Markov chain is called simple random walk with reflecting boundaries. The state space is $\{1, 2, \ldots, n\}$. It is defined as follows:

$$\begin{aligned} P(X_{t+1} = i+1 | X_t = i) &= P(X_{t+1} = i-1 | X_t = i) = 1/2, \ \forall i \in \{2, \ldots, n-1\} \\ P(X_{t+1} = n-1 | X_t = n) &= 1/2 \\ P(X_{t+1} = n | X_t = n) &= 1/2 \\ P(X_{t+1} = 1 | X_t = 1) &= 1/2 \\ P(X_{t+1} = 2 | X_t = 1) &= 1/2. \end{aligned}$$

a) Write code for simulating this Markov chain.

b) Find the limiting distribution when $n = 5$ and $n = 10$ by simulating the chain multiple times starting from $X_0 = 1$. Make a conjecture about the limiting distribution for all n.

c) For $n = 5$, estimate the expected number of steps needed to return to i starting at i for all $i \in \{1, 2, 3, 4, 5\}$.

d) For $n = 5, 6, 7, 8, 9, 10$, estimate the expected number of steps needed to move to state n starting from state 1 in the Markov chain and make a plot. Make a conjecture about the relationship.

Note: for part d), I would like a function that given n, approximates the expected time to move from 1 to n. If you are unsure how to approach such an problem, please come see me for tips once you have simulated some data.