HOMEWORK 3

STA 624.01, Applied Stochastic Processes
Spring Semester, 2007

Due: Monday, February 5th, 2007

Readings: Sections 2.1 of text

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also please box your numerical answers.

This week's exercises focuses on tricks for finding probabilities in Markov chains by modifying the graph. For example, to find the expected value of random variables of the form:

$$T_{ij} := \inf\{t > 0 : X_t = j | X_0 = i\},$$

we'll use a trick.

You already know how to determine the expected time it takes for the chain to return to a state x that is recurrent:

$$E[T_{xx}] = E[R_x] = 1/\pi(x).$$

Now suppose that you are trying to find $E[T_{xy}]$ where $x \neq y$. One method: modify the Markov chain by adding a dummy node z, setting

$$P(X_{t+1} = x | X_t = z) = 1,$$

removing all outgoing edges from y and making

$$P(X_{t+1} = z | X_t = y) = 1.$$

So any path from z to itself moves from z to x, then x to y, then y back to z. So $E[R_z] = 2 + E[T_{xy}]$.

Regular Problems

1 Again consider a Markov chain on states $\{a, b, c\}$ with transition matrix.

$$A = \begin{pmatrix} .2 & .3 & .5 \\ .7 & .1 & .2 \\ .4 & .4 & .2 \end{pmatrix}$$

a) Find $E[R_1]$, $E[R_2]$, and $E[R_3]$
b) Find $E[T_{13}]$, $E[T_{23}]$ and $E[T_{12}]$.

2 (Resnick 1992) The Media Police have identified six states associated with television watching: 0 (never watch TV), 1 (watch only PBS), 2 (watch TV fairly frequenctly), 3 (addict), 4 (undergoing behavior modification), 5 (brain dead). Transitions from state to state can be modelled as a Markov chain with the following transition matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ .1 & 0 & .5 & .3 & 0 & .1 \\ 0 & 0 & 0 & .7 & .1 & .2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Which states are transient and which are recurrent?
(b) Starting from state 1, what is the probability that state 5 is entered before state 0, i.e., what is the probability that a PBS viewer will wind up brain dead?

3 (Problem 1.11 in text) Consider a Markov chain that keeps track of the number of papers in a pile. The state space is $\{0, 1, 2\}$ and the transition matrix is:

$$A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Now let X_n, Y_n be independent realizations of this Markov chain (for instance you can think of this as two piles of papers that evolve independently). Suppose $X_0 = 0$, $Y_0 = 2$ and let

$$T = \inf\{n : X_n = Y_n\},$$

so that T is the first time that the two piles have the same number of papers in them.

(a) Find $E(T)$.

(b) What is $P\{X_T = 2\}$?

(c) In the long run, what percentage of the time are both chains in the same state?

(Hint: Consider the nine-state Markov chain $Z_n = (X_n, Y_n)$, and then for (b) think about altering the chain so that $(0, 0)$, $(1, 1)$, and $(2, 2)$ only move to themselves.

4 Consider the Markov chain described by the transition matrix A in problem 4. Suppose it models the number of papers in a pile of papers.

(a) After a long time, what would be the expected number of papers in the pile?

(b) Assume the pile starts with 0 papers. What is the expected time until the pile will again have 0 papers?

Computer Problems

No simulation this week, but you will need to figure out good ways to construct transition matrices. Check out the command `diag` in MATLAB (recall that you can get help on a command like this by typing `help diag` in MATLAB).

Simple Random Walk Consider simple random walk on $\{0, \ldots, n\}$, so with probability $1/2$ we move to the right and with probability $1/2$ to the left, unless that would make us go below 0 or above n, in which case we stay where we are.

Part (a) Starting at 0, compute exactly the expected number of steps needed to get to n, where n runs from 1 up to 100. (Recall in the last homework you estimated this using simulation, here you are finding it exactly.) Plot this data and conjecture the form of this function.

Note: given data in vectors x and y, if you believe that $y = \exp(ax + b)$, then if you plot $\ln(y)$ versus x, you should get a straight line with slope a and intercept b. The tutorials describe how to use MATLAB to do linear regression. A simpler method is just to print out the plot, get a ruler, and eyeball a best fit line.

If you believe the curve is of the form $y = bx^a$, then plot $\ln(y)$ versus $\ln(x)$. This will be a straight line with slope a and intercept b.

Part (b) The total variation distance between two probability distribution $\mathbf{P}_1$ and $\mathbf{P}_2$ is

$$||\mathbf{P}_1 - \mathbf{P}_2||_{TV} = \sup_{\text{meas}A} \mathbf{P}_1(A) - \mathbf{P}_2(A).$$

If these two probability distributions are over a finite space and are represented by vectors p_1 and p_2, then

$$||p_1 - p_2||_{TV} = (1/2) \sum_{i \in \Omega} |p_1(i) - p_2(i)|.$$

Starting at 0, compute the total variation distance between the distribution of X_t and π, the stationary distribution, after t steps, for t from 1 to 100, and n from 1 to 100. Plot total variation versus time for $n = 1, 10, 50, 100$. Plot total variation versus n for times $t = 1, 10, 50, 100$. Estimate a formula for the total variation distance as a function of n and t.