HOMEWORK 4

STA 624, Applied Stochastic Processes

Due: Friday, February 16th.

Readings: Finish Chapter 2 of text

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$, and $1/80 = .01250$ for example. Also please box your numerical answers.

Regular Problems

1 Gambler's Ruin, part I Suppose that a gambler plays a fair game so that at each play of the game she loses a \$1 with probability 1/2, and gains a \$1 with probability 1/2. The gambler starts with \$32 and stops when she reaches \$100 or is out of money.

a) What is the probability that the gambler ends up with \$100?

b) What is the expected number of plays before the gambler has either \$100 or \$0?

c) Now suppose the gambler starts with \$73. What is the probability that the gambler ends up with \$100?

2 Gambler's Ruin, part II Now suppose that the gambler is betting on red in American roulette. The chance of winning a \$1 is now 18/38 and losing a \$1 is 20/38.

a) What is the probability that the gambler ends up with \$100?

b) What is the expected number of plays before the gambler has either \$100 or \$0?

c) Now suppose the gambler starts with \$73. What is the probability that the gambler ends up with \$100?

The moral of the Gambler's Ruin problem: a tiny change away from a fair game can have an enormous effect on the chances of walking away a winner.

3 SIS model In the SIS model, individuals in a population are either susceptible to infection (S), or infected (I). A susceptible person can become infected, then once the disease has run its course, becomes susceptible again. (So $S \to I \to S$, which is why the model is called what it is.)

Suppose that N is the total population, and the state measures the number of infected people, so $\Omega = \{0, 1, 2, \ldots, N\}$. There are two parameters: β which controls how fast people are infected, and γ which controls how fast people are cured. Let

$$\begin{aligned} P(I_{t+1} = i + 1 | I_t = i) &= \beta i(N - i)/N, \\ P(I_{t+1} = i - 1 | I_t = i) &= \gamma i, \\ P(I_{t+1} = i | I_t = i) &= 1 - \beta i(N - i)/N - \gamma i. \end{aligned}$$

For $N = 20, \beta = .01,, \gamma = .005$, and $I_0 = 2$, find the expected length of the epidemic, that is, the expected number of steps needed before the number of infected people reaches 0.

Computer Problems

Coupling Consider the simple random walk on $\{0, \ldots, n\}$ with partially reflecting boundaries, so add 1 with probability 1/2 and subtract 1 with probability 1/2, unless that would take you outside the state space, in which case stay where you are.

One coupling for processes X_t and Y_t for this chain is to use the same random stream $U_0, U_1, U_2, \ldots$ to generate both $\{X_t\}_{t=0}^{\infty}$ and $\{Y_t\}_{t=0}^{\infty}$.

For $n = 3, n = 6, n = 12$ and $n = 24$, and $t \in \{1, 2, \ldots, n^2\}$ do the following. 1) Find the total variation distance between $\mathcal{L}(X_t | X_0 = 0)$ and the stationary distribution. 2) Estimate the probability that X_t and Y_t

have not coupled by time t where $X_0 = 0$ and Y_0 is a draw from the stationary distribution by simulating the Markov chain.

Plot these two quantities versus t for each n, so you will end up with four plots.