HOMEWORK 5

STA 624, Applied Stochastic Processes

Due: Friday, February 23rd.

Readings: Finish Chapter 2 of text

Regular Problems

1 (Lawler 2.2) Consider the following Markov chain with state space $\Omega = \{0, 1, \ldots\}$. A sequence of positive numbers $p_1, p_2, \ldots$ is given with $\sum_{i=1}^{\infty} p_i = 1$. Whenever the chain reaches state 0 it chooses a new state according to the p_i. Whenever the chain is at a state other than 0, it proceeds deterministically, one step at a time, toward 0. In other words, the chain has transition probabilities:

$$p(x, x-1) = 1, \; x > 0,$$

$$p(0, x) = p_x, \; x > 0.$$

This is a recurrent chain since the chain keeps returning to 0. Under what conditions on the p_x is the chain positive recurrent? In this case, what is the limiting probability distribution π? [Hint: it may be easier to compute $E(T)$ directly where T is the time of the first return to 0 starting at 0.]

2 (Lawler 2.3) Consider the Markov chain with state space $\Omega = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p(x, x+1) = 2/3; \; p(x, 0) = 1/3.$$

Show that the chain is positive recurrent and give the limiting probability π.

3 (Lawler 2.4) Consider the Markov chain with state space $\Omega = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p(x, x+2) = p, \; p(x, x-1) = 1-p, \; x > 0.$$

$$p(0, 2) = p, \; p(0, 0) = 1-p.$$

For which values of p is this a transient chain?

4 Back to finite state space Markov chains. This is a method for ordering inventory called the s/S method. Suppose that demands of product for each day are found by an i.i.d. stream of random variables $D_1, D_2, \ldots$. Let X_t be the amount of inventory in stock at the beginning of each day. If $X_t \leq s$, then immediately the owner runs out and buys more items to bring the inventory up to level S. Then D_t items are sold during the course of the day.

Some notation: let $x_+ = x$ if $x > 0$, and 0 if $x \leq 0$ (in formula form, $x_+ = (1/2)[|x| + x]$). Then we have

$$X_{n+1} = \begin{cases} (X_n - D_{n+1})_+, & \text{if } s < X_n \leq S \\ (S - D_{n+1})_+ & \text{if } X_n \leq s \end{cases}$$

Since the demands $\{D_i\}$ are themselves functions of our ultimate source of randomness $U_1, U_2, \ldots$, this is a Markov chain.

Two of the things we could be interested in are "long run average stock level", defined as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} X_j = \sum_{i=0}^{S} i\pi(\{i\}),$$

and "long run fraction of periods with unsatisfied demand"

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1(((D_n > X_n) \text{ and } (X_n > s)) \text{ or } ((D_n > S) \text{ and } (X_n \le s))).$$

Now for the actual problem. Suppose that $s = 2$ and $S = 6$, and demand each day has the following distribution:

$$P(D_1 = 0) = .4,\ P(D_1 = 1) = .2,\ P(D_1 = 2) = .3,\ P(D_1 = 3) = .1.$$

Find the long run average stock level, and the long run fraction of periods with unsatisfied demand.

Computer Problems

Simple Random Walk in $\mathbf{Z}^d$ Consider simple random walk on $\mathbf{Z}^1$, $\mathbf{Z}^2$, and $\mathbf{Z}^3$. For each of these, start at the origin and do the following. Estimate the expected distance aways from the origin after t steps, for t running from 1 to 100. Just use Euclidean distance, so in $\mathbf{Z}^3$, the distance of point (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$. Conjecture a formula for this expected distance for $d = \{1, 2, 3\}$.