HOMEWORK 6

STA 624, 2007 Applied Stochastic Processes

Due: Monday, March 5th, 2007.

Readings: Section 3.1 to 3.2

Regular Problems For the following problems, let p_i be the probability that a member of the population has i children at the next step.

1 (Lawler, 2.6) Given a branching process with the following offspring distributions, determine the extinction probability a.

(a) $p_0 = .25, p_1 = .4, p_2 = .35$.
(b) $p_0 = .5, p_1 = .1, p_3 = .4$.
(c) $p_0 = .91, p_1 = .05, p_2 = 0.01, p_3 = 0.01, p_6 = 0.01, p_{13} = .01$.
(d) $p_i = (1 - q)q^i$, for some $0 < q < 1$.

2 (Lawler 2.7) Consider the branching process with $p_0 = .5, p_1 = .1, p_3 = .4$ and suppose $X_0 = 1$.

(a) What is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not die out in the first generation ($X_1 > 0$)?

(b) What is the probability that the population is extinct in the third generation, given that it was not extinct in the second generation?

3 (Lawler 2.9) You'll want to use MATLAB for this one:

(a) Suppose $p_0 = p_1 = p_2 = 1/3$. Find the probability that the population dies out in 20, 200, and 2000 steps.

(b) Repeat part (a) with $p_0 = .35, p_1 = .33, p_2 = .32$

4 (Lawler 2.15 (a)) Let X_1, $X_2, \cdots$ be independent identically distributed random variables taking values in the integers with mean 0. Let $S_0 = 0$ and

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Let

$$G_n(x) = E\left[\sum_{j=0}^{n} 1_{\{S_j = x\}}\right]$$

be the expected number of visits to x in the first n steps. Show that for all n and x, $G_n(0) \geq G_n(x)$. (Hint: consider the first j with $S_j = x$.)

Computer Problem: Recurrence properties of two-dimensional lattices In lecture we showed that for a two dimension grid lattice simple random walk is null recurrent. Now we explore the behavior of simple random walk on a triangular lattice and a hexagonal lattice through simulation.

In the triangular lattice, we tile the plane using equilateral triangles, in the hexagonal lattice, we tile the plane using regular hexagons. Check out
`http://library.thinkquest.org/16661/simple.of.regular.polygons/regular.1.html?tqskip1=1&tqtime=0205`

for pictures of what these tilings (a.k.a. tesselations) look like. The grid lattice we studied in class is just the tiling with squares. There we discovered that the chain was null recurrent, so starting at the origin, the probability of return was 1.

(a) For the triangular lattice, make a conjecture about whether the chain is recurrent or transient. Let T be the time needed to return to the origin. Use two approaches to test recurrence. First plot estimates of $P(T > n)$ for large enough values of n to see if this approaches 0 or not. Second, estimate

$$P(X_n = \text{origin} | X_0 = \text{origin})$$

for a variety of values of n, and conjecture a relationship between these values and n. Use this conjecture to show that the expected number of visits to the origin is either finite or infinite.

(b) Repeat part (a) for a hexagonal lattice.

Note: In representing the current position in hexagonal and triangular lattices, it is easiest to just use two coordinates as in the square grid case. Use the first coordinate to represent number of moves "up and to the right" and the second coordinate to represent moves "to the right". The points of the hexagonal lattice are a subset of the points in the triangular lattice, so this framework can be used for both situations.