# A review of DTMC and CTMC

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# Some definitions on MC

**Definition** A *discrete time stochastic process* is a collection of random variables $\{X_0, X_1, X_2, \ldots\}$ defined on a common sample space and state space.

**Definition** A (time homogeneous) *Markov chain* is a stochastic process such that for all $t \in \{0, 1, \ldots\}$ and measurable sets $A$,

$$P(X_{t+1} \in A | X_0, X_1, \ldots, X_t = x) = P(X_{t+1} \in A | X_t = x).$$

[All Markov chains in this course will be time homogeneous unless explicitly stated otherwise.]

# Finite State MC

**Definition** Given a Markov chain with finite state space $\Omega$, a *transition matrix* is a matrix whose entry in the $i$th row and $j$th column is

$$P(X_{t+1} = j | X_t = i).$$

**Definition** Given a Markov chain with finite state space $\Omega$, the *transition graph* has vertex set $\Omega$, and has directed edges $(i, j)$ with weight

$$P(X_{t+1} = j | X_t = i)$$

whenever this weight is positive.

**Definition** $\pi$ is a *stationary* or *invariant* distribution for a Markov chain if $X_t \sim \pi$ implies that $X_{t+1} \sim \pi$ (i.e. $\sum_{y \in \Omega} \pi_y p(y, x) = \pi_x$).

**Definition** For countable state Markov chains, if

$$\sum_{x \in \Omega} \pi_x P(X_{t+1} = y | X_t = x) = \pi_y$$

then $\pi$ is a stationary distribution. These are called the *balance equations.*

**Definition** $\pi$ is a *limiting* distribution for a countable state Markov chain if

$$\lim_{t \to \infty} P(X_t = i | X_0 = j) = \pi_i,$$

for all states $i$ and $j$.

**Definition** States $x$ and $y$ of a Markov chain communicate if for some $n$ and some $m$, $P(X_n = y|X_0 = x) > 0$ and $P(X_m = x|X_0 = y) > 0$.

**Definition** A Markov chain is *irreducible* if all states communicate. Otherwise it is reducible.

**Definition** The *period* of an irreducible Markov chain is

$$\text{gcd}\{n : P(X_n = x|X_0 = x) > 0\}$$

for any state $x$. A chain with period 1 is called *aperiodic*.

# Ergodic Theorem for finite state Markov chains

For irreducible aperiodic finite state Markov chains, a unique stationary distribution exists with $\pi_x > 0$ for all $x \in \Omega$, and this will also be the limiting distribution. In addition, the expected time of return from $x$ to itself will be $1/\pi_x$.

# Countable State MC

**Definition** Let

$$R_x = \inf\{t > 0 : X_t = x | X_0 = x\}$$

be the *return time* for the state $x$. Then state $x$ is *recurrent* if $P(R_x < \infty) = 1$. If any state in an irreducible chain is recurrent, they all are, and it is a *recurrent chain.*

**Definition** A state that is not recurrent is *transient*. An irreducible chain which is not recurrent is a *transient chain*.

**Fact** A state $x$ is transient iff $E(N(x)|X_0 = x) < \infty$ where $N(x) = \sum_{m=1}^{\infty} 1_{\{X_m = x\}}$.

**Definition** Consider a chain that is recurrent and aperiodic. If there exist states $x$ and $y$ in $\Omega$ such that $\lim_{n\to\infty} P(X_n = y|X_0 = x) = 0$, the chain is *null recurrent*. When the limit is positive, it is *positive recurrent*.

**Fact** Let $R$ be the time needed to return to state $x$ given $X_0 = x$. For null recurrent chains, $E(R) = \infty$ and $P(R < \infty) = 1$.

# Ergodic Thm for countable state space

For an irreducible aperiodic positive recurrent chain on a countable state space, there exists a unique stationary distribution $\pi$ with $\pi_i > 0$. Also, $\pi$ is the limiting distribution, and if $R$ is the time for return to $x$ starting from $x$, $E(R) = 1/\pi_x$. If the chain is null recurrent or transient, there is no stationary distribution $\pi$.

**Definition** A *branching process* is a special Markov chain on $\{0, 1, 2, \ldots\}$ such that if $\xi_1^n, \xi_2^n, \ldots$ are i.i.d. draws from a distribution on the nonnegative integers with a mean $\mu$, then

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^n.$$

**Theorem**: If $\mu \leq 1$ and $p_0 > 0$ then the extinction probability equal to 1. If $\mu > 1$ then the extinction probability is less than 1 and equal to the smallest positive root of $t = \phi(t)$ with $0 < t < 1$.

# Continuous Time Markov chains

**Definition** A *continuous time stochastic process* is a collection $\{Z_\alpha\}$ of random variables from a common sample space to a state space, where $\alpha \in \mathbf{R}^n$.

**Definition** A stochastic process $\{Y_t\}$ has the *Markov property* if for all measurable $A$, and $0 < s < t$

$$P(Y_t \in A | Y_r, 0 \leq r \leq s) = P(Y_t \in A | Y_s).$$

**Definition** A *continuous time Markov chain* is a continuous time stochastic process with the Markov property that changes value at a countable number of times, $0 = T_0 < T_1 < T_2 < \ldots$ with probability 1.

**Definition** For finite state space continuous time Markov chains, the *infinitesimal generator* of the chain is an $|\Omega| \times |\Omega|$ matrix with $A(i,j) = \alpha(i,j)$ for $j \neq i$, and $A(i,i) = -\sum_{j \neq i} \alpha(i,j)$.

**Fact** For $\{X_t\}$ a continuous time Markov chain with jumps at $0 = T_0 < T_1 < \ldots$,
1) $T_i - T_{i-1}$ are independent random variables, and $\{T_i - T_{i-1} | X_{T_{i-1}}\}$ is an exponential random variable with rate that only depends on $X_{T_{i-1}}$, and
2) $Y_i = X_{T_i}$ is a discrete time Markov chain called the *underlying discrete time chain* or *embedded chain* or *jump chain*.

**Definition** *Recurrence, transience, positive recurrence, and null recurrence* are defined the same way as for discrete time Markov chains. A continuous time Markov chain is *irreducible* if for all $x, y \in \Omega$ and $t > 0$:

$$P(X_t = y | X_0 = x) > 0.$$

**Fact** A continuous time Markov chain is irreducible, recurrent, or transient if and only if the underlying discrete chain is (respectively) irreducible, recurrent, or transient. However, one can be positive recurrent while the other is null recurrent.

# Ergodic Thm for continuous time

**Definition** Define the return time to a state $x$ in a continuous time Markov Chain as follows:

$$R_x = \inf\{t > T_1 : X_t = x | X_0 = x\}.$$

**Ergodic Thm for continuous time** For an irreducible positive recurrent continuous time Markov chain on a countable state space, there exists a unique stationary distribution $\pi$ with $\pi(i) > 0$. Also, $\pi$ is the limiting distribution, and if $R$ is the time for return to $x$ starting from $x$, $E(R) = -1/[A(x,x)\pi(x)]$. If the chain is null recurrent or transient, there is no stationary distribution $\pi$.

# Poisson process

**Definition** A *Poisson process* is a process $\{X_t\}$ satisfying:

- $X_0 = 0$.
- The number of events during one time interval does not affect the number of events during a different time interval.
- The average rate at which events occur remians constant.
- Events occur once at a time.

# Birth death chain

**Definition** A *birth death chain* is a continuous time Markov chain on state space $\{0, 1, 2, \ldots\}$ where $\alpha(i, i+1) = \lambda_i$ for $i \geq 0$, $\alpha(i, i-1) = \mu_i$ for all $i \geq 1$ and no other edges exist.

**Fact** A *Poisson process* is a birth death chain where $\lambda_i = \lambda$ and $\mu_i = 0$ for all $i$.

**Definition** The *Yule process* is the process cross between Poisson process and Branching process. Each individual present at time $t$ splits into 2 during the time interval $(t, t + \Delta t)$ with probability $\lambda \Delta t + o(\Delta t)$ and $\mu = 0$.

# Birth and death process

**Theorem** A birth death chain is transient if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} < \infty.$$

# Birth and death process

## Theorem

A birth death chain is positive recurrent if and only if

$$q = \sum_{n=0}^{\infty} \frac{\lambda_0 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

(the term $n = 0$, the sum is equal to 1). Also the stationary distribution $\pi$ is

$$\pi_i = \frac{\lambda_0 \lambda_2 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \cdot q^{-1}.$$

# Chapman-Kolmogorov forward equations

**Theorem** $\alpha(x,y)$ is the rate at which a continuous time Markov chain moves from $x$ to $y$ and if $\alpha(x) = \sum_{y \neq x} \alpha(x,y)$, then

$$\frac{d}{dt} p_t(x,y) = -\alpha(y) p_t(x,y) + \sum_{z \neq y} \alpha(z,y) p_t(x,z),$$

where $p_t(x,y) = P(X_t = y | X_0 = x) = P(X_{t+s} = y | X_s = x)$.

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where $p_t(x, y) = P(X_t = y | X_0 = x) = P(X_{t+s} = y | X_s = x)$.

# Balance equations

**Definition** If $\alpha(x, y)$ is the rate at which a continuous time Markov chain moves from $x$ to $y$, then the *balance equations* are for all $x \in \Omega$:

$$\pi(x) \sum_{y \neq x} \alpha(x, y) = \sum_{y \neq x} \pi(y)\alpha(y, x).$$

The *detailed balance equations* (aka *reversibility*) are for all $x, y \in \Omega$:

$$\pi(x)\alpha(x, y) = \pi(y)\alpha(y, x).$$

# $M/M/1$ **queue**

This is a birth and death process with $\lambda_n = \lambda$ and $\mu_n = \mu$.

$\sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^n < \infty$ iff it is transient. Thus $\mu < \lambda$ iff it is transient.

$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n < \infty$ iff it is positive recurrent. Thus $\lambda < \mu$ iff it is positive recurrent.

If $\lambda < \mu$ then the stationary distribution $\pi(n) = (1-\rho)\rho^n$ where $\rho = \lambda/\mu$.

The expected value of the length of the queue is $\lambda/(\mu - \lambda)$ if $\lambda < \mu$.

# $M/M/k$ queue

This is a birth and death process with $\lambda_n = \lambda$ and

$$\mu_n = \begin{cases} k\mu & \text{if } n \geq k \\ n\mu & \text{if } n < k. \end{cases}$$

$\sum_{n=1}^{\infty} C \left(\frac{k\mu}{\lambda}\right)^n < \infty$ iff it is transient. Thus it is transient iff $k\mu < \lambda$.

$\sum_{n=0}^{\infty} C \left(\frac{\lambda}{k\mu}\right)^n < \infty$ iff it is positive recurrent. Thus $\lambda < k\mu$ iff it is positive recurrent.

# $M/M/\infty$ queue

$M/M/\infty$ is never transient and always positive recurrent.

The stationary distribution is

$$\pi(n) = \frac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!}.$$

# Reversible MC

**Definition** A discrete time Markov chain is called *reversible* with respect to a distribution $\pi$ if

$$\pi(x)p(x, y) = \pi(y)p(y, x), \forall x,\ y \in \Omega.$$

**Definition** A continuous time Markov chain is called *reversible* with respect to a distribution $\pi$ if

$$\pi(x)\alpha(x, y) = \pi(y)\alpha(y, x), \forall x,\ y \in \Omega.$$

**Definition** A discrete time Markov chain is called *symmetric* if

$$p(x, t) = p(y, x), \; \forall x, \; y \in \Omega.$$

**Definition** A continuous time Markov chain is called *symmetric* if

$$\alpha(x, t) = \alpha(y, x), \; \forall x, \; y \in \Omega.$$

# Time Reversible MC

**Definition** A discrete time Markov chain $\{X_n\}$ is called *time reversed* Markov chain of $\{Y_n\}$ if

$$X_n = Y_{-n},\, n \in \mathbb{Z}.$$

**Definition** A continuous time Markov chain $\{X_t\}$ is called *time reversed* Markov chain of $\{Y_t\}$ if

$$X_t = Y_{-t},\, t \in \mathbb{R}.$$

# Time Reversible MC

**Definition** A discrete time Markov chain $\{X_n\}$ is called *time reversible* if

$$p_r(x, y) = p(x, y), \ \forall x, \ y \in \Omega,$$

where $p_r(x, y)$ is a transition probability from $x$ to $y$ in its time reversed Markov chain.

**Definition** A continuous time Markov chain $\{X_n\}$ is called *time reversible* if

$$\alpha_r(x, y) = \alpha(x, y), \ \alpha_r(x) = \alpha(x), \forall x, \ y \in \Omega,$$

where $\alpha_r(x, y)$ is a rate of change from $x$ to $y$ in its time reversed Markov chain.