Review for first midterm

STA 624, Applied Stochastic Processes

In this guide, I've put editorial comments in brackets [like this]. The bracketed comments are not part of the definitions, but are background or extra material. This is not intended to cover everything we've talked about in the course, but the most important elements.

1 Discrete time Markov chains

[See the probability review for definitions of random variable, sample space, and state space.]

Definition A *discrete time stochastic process* is a collection of random variables $\{X_0, X_1, X_2, \ldots\}$ defined on a common sample space and state space.

Definition A (time homogeneous) *Markov chain* is a stochastic process such that for all $t \in \{0, 1, \ldots\}$ and measurable sets A,

$$P(X_{t+1} \in A | X_0, X_1, \ldots, X_t = x) = P(X_{t+1} \in A | X_t = x).$$

[All Markov chains in this course will be time homogeneous unless explicitly stated otherwise.]

Definition Given a Markov chain with finite state space Ω, a *transition matrix* is a matrix whose entry in the ith row and jth column is

$$P(X_{t+1} = j | X_t = i).$$

Definition Given a Markov chain with finite state space Ω, the *transition graph* has vertex set Ω, and has directed edges (i, j) with weight $P(X_{t+1} = j | X_t = i)$ whenever this weight is positive.

Definition π is a *stationary* or *invariant* distribution for a Markov chain if $X_t \sim \pi$ implies that $X_{t+1} \sim \pi$.

Definition For countable state Markov chains, if

$$\sum_{x \in \Omega} \pi_x P(X_{t+1} = y | X_t = x) = \pi_y$$

then π is a stationary distribution. These are called the *balance equations.*

Definition π is a *limiting* distribution for a countable state Markov chain if

$$\lim_{t \to \infty} P(X_t = i | X_0 = j) = \pi_i,$$

for all states i and j.

Definition States x and y of a Markov chain communicate if for some n and some m, $P(X_n = y | X_0 = x) > 0$ and $P(X_m = x | X_0 = y) > 0$.

Definition A Markov chain is *irreducible* if all states communicate. Otherwise it is reducible.

Definition The *period* of an irreducible Markov chain is

$$\gcd\{n : P(X_n = x | X_0 = x) > 0\}$$

for any state x. A chain with period 1 is called *aperiodic.*

Ergodic Theorem for finite state Markov chains For irreducible aperiodic finite state Markov chains, a unique stationary distribution exists with $\pi_x > 0$ for all $x \in \Omega$, and this will also be the limiting distribution. In addition, the expected time of return from x to itself will be $1/\pi_x$.

Definition Let

$$R_x = \inf\{t > 0 : X_t = x | X_0 = x\}$$

be the *return time* for the state x. Then state x is *recurrent* if $P(R_x < \infty) = 1$. If any state in an irreducible chain is recurrent, they all are, and it is a *recurrent chain.*

Definition A state that is not recurrent is *transient*. An irreducible chain which is not recurrent is a *transient chain*.

Definition Consider a chain that is recurrent and aperiodic. If there exist states x and y in Ω such that $\lim_{n\to\infty} P(X_n = y|X_0 = x) = 0$, the chain is *null recurrent*. When the limit is positive, it is *positive recurrent*.

Fact Let R be the time needed to return to state x given $X_0 = x$. For null recurrent chains, $E(R) = \infty$ and $P(R < \infty) = 1$.

Ergodic Thm for countable state space For an irreducible aperiodic positive recurrent chain on a countable state space, there exists a unique stationary distribution π with $\pi_i > 0$. Also, π is the limiting distribution, and if R is the time for return to x starting from x, $E(R) = 1/\pi_x$. If the chain is null recurrent or transient, there is no stationary distribution π.

Definition A *branching process* is a special Markov chain on $\{0, 1, 2, \ldots\}$ such that if $\xi_1^n, \xi_2^n, \ldots$ are i.i.d. draws from a distribution on the nonnegative integers with a mean μ, then

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^n.$$

Theorem: If $\mu \leq 1$ and $p_0 > 0$ then the extinction probability equal to 1. If $\mu > 1$ then the extinction probability is less than 1 and equal to the smallest positive root of $t = \phi(t)$ with $0 < t < 1$.