HOMEWORK 2

STA 624.01, Applied Stochastic Processes
Spring Semester, 2008

Due: Friday, January. 25, 2008

Readings: Chapter 1 of text, Tutorial 2 on MATLAB

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 Consider the telecommunications Markov chain with transition matrix:

$$A = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

where $0 \leq p, q \leq 1$.

Suppose you flip a fair coin to start the chain in the 0 or 1 position.

a) What is the probability vector for X_0?

b) What is the distribution of X_1 (the chain after one step)?

c) What is the distribution of X_2 (the chain after two steps)?

d) Compute $P(X_{20} = 0 | X_0 = 0)$ if $p = 1/3$ and $q = 1/4$.

e) Compute $P(X_{10} = 1, X_{15} = 0 | X_0 = 1)$, if $p = 1/3$ and $q = 1/4$.

2 Let X_n be a Markov chain on state space $\{1, 2, 3, 4, 5\}$ with a transition matrix:

$$P = \begin{pmatrix} 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

What is the probability of realization $X_0 = 1, X_1 = 2, X_2 = 5, X_3 = 1, X_4 = 1$ if the initial state distribution is uniform over the state space of this Markov chain?

3 Consider two urns A and B containing N balls. An experiment is performed in which one of the N balls is selected with probability depending on the urn contents (i.e., if A currently has k balls, a ball is chosen from A with probability k/N or from B with probability $(N-k)/N$). Then, an urn is selected and then depositing the selected ball in the selected urn. Urn A is chosen with probability k/N or urn B is chosen with probability $(N-k)/N$. Determine the transition matrix of the Markov chain with states represented by the contents of A.

4 A sequence of electrical impulses passes a measurement instrument that stores the largest value measured so far. Assume that the impulses at time points $0, 1, 2, 3, \cdots$ can be modelled as independent random variables $Y_0, Y_1, Y_2, Y_3, \cdots$ with a uniform distribution on $\{1, 2, 3, 4, 5\}$. Thus, if $X_1, X_2, X_3, \cdots$ are the values stored at time points $0, 1, 2, 3, \cdots$, then

$$X_n = \max(Y_0, Y_1, Y_2, \cdots, Y_n) \text{ for } n = 0, 1, 2, 3, \cdots.$$

Motivate that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain and write down the transition probability matrix.

Computer Problems

For this problem, please print out all code used and all results.

This Markov chain is called simple random walk with reflecting boundaries. The state space is $\{1, 2, \ldots, n\}$. It is defined as follows:

$$\begin{aligned}
P(X_{t+1} = i+1 | X_t = i) &= p, \ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = i-1 | X_t = i) &= 1-p, \ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = n-1 | X_t = n) &= 1-p \\
P(X_{t+1} = n | X_t = n) &= p \\
P(X_{t+1} = 1 | X_t = 1) &= 1-p \\
P(X_{t+1} = 2 | X_t = 1) &= p.
\end{aligned}$$

a) Write code for simulating this Markov chain.

b) Find the limiting distribution when $n = 5$ and $n = 10$ for $p = 0.33$, 0.5, 0.66 by simulating the chain multiple times starting from $X_0 = 1$.

c) For $n = 5$ with $p = 0.33$, 0.5, 0.66, estimate the expected number of steps needed to return to i starting at i for all $i \in \{1, 2, 3, 4, 5\}$.