HOMEWORK 2

STA 624.01, Applied Stochastic Processes
Spring Semester, 2008

Due: Friday, Feburuary 1st, 2008

Readings: Chapter 1 of text

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 A simple model of DNA is that the sequence of the bases A, C, G and T forms a Markov chain. Assume this chain has transition probability matrix

$$\begin{pmatrix} 0.30 & 0.22 & 0.21 & 0.27 \\ 0.28 & 0.22 & 0.30 & 0.20 \\ 0.23 & 0.32 & 0.23 & 0.22 \\ 0.18 & 0.22 & 0.20 & 0.30 \end{pmatrix}.$$

here each row and column corresponds to A, C, G, T, respectively. A subsequence that has been analysed reads ATGxxCGT, where "xx" means that one is uncertain about these two bases. Biological considerations however suggest that these two symbols are either AC ot TG. Which of these two alternatives is the most probable?

2 A Markov chain with states $S = \{1, \cdots, 4\}$ has transition probability matrix:

$$P = \begin{pmatrix} 0.3 & 0 & 0 & 0.7 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Which states are recurrent and which are transient?
(b) Compute the periods of states 2 and 4, respectively.

3 (a) Construct an irreducible Markov chain with 3 states and period 2. Give your answer as a transition diagram.

(b) A Markov chain has transition probability matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

Compute the stationary distribution π and find out if $P_n \to \pi$ as $n \to \infty$, independently of the initial distribution.

4 A stochastic process in discrete time and with discrete state space is said to be a Markov chain of order m if

$$\begin{aligned} & P(X_{n+1} = i_{n+1} | X_n = i_n,\, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) \\ = \; & P(X_{n+1} = i_{n+1} | X_n = i_n, \cdots, X_{n-m+1} = i_{n-m+1}) \end{aligned}$$

for all $n \geq m-1$ and $i_0, i_1, \cdots, i_{n+1} \in S$. In other words, the next value X_{n+1} depends (stochastically) on the m previous values $X_n, \cdots, X_{n-m+1}$. An ordinary Markov chain is thus a Markov chain of order 1. Consider a Markov chain of order 2 with transition probabilities $p_{i,j;k} = P(X_{n+1} = k | X_n = i,\, X_{n-1} = j)$ for $i, j, k \in \{0, 1\}$ given by the following table:

i	j	k	$p_{i,j;k}$
0	0	0	0.4
0	0	1	0.6
0	1	0	0.8
0	1	1	0.2
1	0	0	0.5
1	0	1	0.5
1	1	0	0.3
1	1	1	0.7

Define the random variable Z_k as the couple $Z_k = (X_k,\, X_{k-1})$. Then $\{Z_k\}$ is an ordinary (order 1) Markov chain with state space $\{(0,0),\, (0,1),\, (1,0),\, (1,1)\}$ (you do not need to prove this).

(a) What is the transition probability matrix P of $\{Z_k\}$?

(b) Compute the limiting probability π.

Computer Problems

Random walk with reflecting boundaries

For this problem, please print out all code used and all results.

Recall we did a simulation with a random walk with reflecting boundaries such that: The state space is $\{1, 2, \ldots, n\}$. It is defined as follows:

$$\begin{aligned}
P(X_{t+1} = i+1 | X_t = i) &= p,\ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = i-1 | X_t = i) &= 1-p,\ \forall i \in \{2, \ldots, n-1\} \\
P(X_{t+1} = n-1 | X_t = n) &= 1-p \\
P(X_{t+1} = n | X_t = n) &= p \\
P(X_{t+1} = 1 | X_t = 1) &= 1-p \\
P(X_{t+1} = 2 | X_t = 1) &= p.
\end{aligned}$$

Using the code you implemented do the following simulation:

a) For $n = 5, 6, 7, 8, 9, 10$ and for $p = 0.33, 0.5, 0.66$, estimate the expected number of steps needed to move to state n starting from state 1 in the Markov chain and make a plot.

b) For $n = 5, 6, 7, 8, 9, 10$ and for $p = 0.33, 0.5, 0.66$, estimate the expected number of steps needed to move to state n starting from state 3 in the Markov chain and make a plot.

c) If you change the transition probability at the boundaries such that:

$$\begin{aligned}
P(X_{t+1} = n-1 | X_t = n) &= 0 \\
P(X_{t+1} = n | X_t = n) &= 1 \\
P(X_{t+1} = 1 | X_t = 1) &= 1 \\
P(X_{t+1} = 2 | X_t = 1) &= 0.
\end{aligned}$$

For $n = 5, 6, 7, 8, 9, 10$ and for $p = 0.33, 0.5, 0.66$, estimate the expected number of steps needed to move to state n or 0 starting from state $\lfloor n/2 \rfloor$ in the Markov chain and make a plot.