HOMEWORK 4

STA 624.01, Applied Stochastic Processes
Spring Semester, 2008

Due: Friday, February 8th, 2008

Readings: Sections 2.1 of text

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also please box your numerical answers.

This week's exercises focuses on tricks for finding probabilities in Markov chains by modifying the graph. For example, to find the expected value of random variables of the form:

$$T_{ij} := \inf\{t > 0 : X_t = j | X_0 = i\},$$

we'll use a trick.

You already know how to determine the expected time it takes for the chain to return to a state x that is recurrent:

$$E[T_{xx}] = E[R_x] = 1/\pi(x).$$

Now suppose that you are trying to find $E[T_{xy}]$ where $x \neq y$. One method: modify the Markov chain by adding a dummy node z, setting

$$P(X_{t+1} = x | X_t = z) = 1,$$

removing all outgoing edges from y and making

$$P(X_{t+1} = z | X_t = y) = 1.$$

So any path from z to itself moves from z to x, then x to y, then y back to z. So $E[R_z] = 2 + E[T_{xy}]$.

Regular Problems

1 Again consider a Markov chain on states $\{a, b, c\}$ with transition matrix.

$$A = \begin{pmatrix} .2 & .3 & .5 \\ .7 & .1 & .2 \\ .4 & .4 & .2 \end{pmatrix}$$

a) Find $E[R_1]$, $E[R_2]$, and $E[R_3]$
b) Find $E[T_{13}]$, $E[T_{23}]$ and $E[T_{12}]$.

2 (Resnick 1992) The Media Police have identified six states associated with television watching: 0 (never watch TV), 1 (watch only PBS), 2 (watch TV fairly frequenctly), 3 (addict), 4 (undergoing behavior modification), 5 (brain dead). Transitions from state to state can be modelled as a Markov chain with the following transition matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ .1 & 0 & .5 & .3 & 0 & .1 \\ 0 & 0 & 0 & .7 & .1 & .2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Which states are transient and which are recurrent?

(b) Starting from state 1, what is the probability that state 5 is entered before state 0, i.e., what is the probability that a PBS viewer will wind up brain dead?

3 Random walk around a circle. Consider the state space $\{0, 1, \ldots, 7\}$. Let $X_1, X_2, \ldots$ be successive rolls of a fair 6-sided die. Let

$$S_n = S_{n-1} + X_n - 8 \cdot 1(S_{n-1} + X_n \geq 8).$$

a) Write out the transition matrix for this transition function.
b) What is the limiting probability that the chain is in state 7?

4 Let $X_1, X_2, \ldots$ be successive rolls of a fair 6-sided die. Let $S_n = X_1 + X_2 + \cdots X_n$. Let

$$T_2 = \min\{n \geq 1 : S_n - 1 \text{ is divisible by } 8\}.$$

Find $E(T_2)$. (Hint: consider the Markov chain from problem 3 of homework 4.)

Computer Problems

Simple Random Walk Consider simple random walk on $\{0, \ldots, n\}$, so with probability 1/2 we move to the right and with probability 1/2 to the left, unless that would make us go below 0 or above n, in which case we stay where we are.

The total variation distance between two probability distribution $\mathbf{P}_1$ and $\mathbf{P}_2$ is

$$||\mathbf{P}_1 - \mathbf{P}_2||_{TV} = \sup_{\text{meas} A} \mathbf{P}_1(A) - \mathbf{P}_2(A).$$

If these two probability distributions are over a finite space and are represented by vectors p_1 and p_2, then

$$||p_1 - p_2||_{TV} = (1/2) \sum_{i \in \Omega} |p_1(i) - p_2(i)|.$$

Starting at 0, compute the total variation distance between the distribution of X_t and π, the stationary distribution, after t steps, for t from 1 to 100, and n from 1 to 100. Plot total variation versus time for $n = 1, 10, 50, 100$. Plot total variation versus n for times $t = 1, 10, 50, 100$. Estimate a formula for the total variation distance as a function of n and t.