HOMEWORK 9

STA 624.01, Applied Stochastic Processes
Spring Semester, 2008

Due: Friday, March 28th, 2008

Readings: Chapter 3 of text

Regular Problems

1 (Lawler 3.5) Let X_t be a Markov chain with state space $\{1,2\}$ and rates $q_{1,2} = 1$, $q_{2,1} = 4$. Let

$$P_t(i,j) = P(Y_t = j | Y_0 = i).$$

Find the matrix P_t.

2 (Lawler 3.8) Consider the continuous-time Markov chain with the state space $\{1,2,3,4\}$ and infinitesmal generator

$$Q = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find the stationary distribution $\bar{\pi}$.

(b) Suppose that the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

(c) Again suppose that the chain starts in state 1. What is the expected amount of time until it changes to state 4?

3 Bernoulli-Laplace model of diffusion: Long before the invention of the term Markov chain, discrete time Markov chains were being studied, they just weren't called that. In the BL model of diffusion, two urns each contain m balls (so $2m$ balls total). A certain number of the balls, b are blue, and the rest $(2m - b)$ are green. Say $b \leq m$.

At each time step, pick one ball uniformly from each urn, and interchange them. Let X_t be the number of blue balls in the first urn.

(a) Find the transition probabilities for this Markov chain.

(b) Find the stationary distribution for this Markov chain.

4 Birth and Death processes: Find an application of Birth and Death processes in your interested field. Summarize how it is applied. What are birth rates and death rates? What is the stationary distribution, if it is known? Why do they use Birth and Death processes?

Computer Problem This computer problem has three parts. In the first part we will look at the Poisson process, in the second we will look at the Poisson distribution, and in the third we will examine the relationship between the Poisson distribution and the binomial distribution.

(a) The Poisson distribution is often used to model the number of events that occur in a fixed time interval or fixed spatial region when they are occurring "at random", like shooting stars.

Suppose you spend five minutes watching shooting stars, and you know that on average they fall at 3 per minute. Thus, the rate parameter for our five-minute time period is lambda=15. We expect, on average, to see 15 shooting stars per five minutes. Show what a Poisson process might look like by using a simulation by breaking that five-minute-long time interval into very small intervals of 1 second and plot them.

(b) Simulate 10,000 star-gazing five-minute periods all at once. Then estimate the count of how many shooting stars you would see in a five-minute stretch. ALso estimate its mean, standard deviation, and variance.

(c) In class we learned that Binomial(n, p) can be approximated by Poisson(λ) with $\lambda = np$, when n is very large. Let us check the approximation. Try different values of $n = 75$ and $n = 100$ with the same λ in (a). See how the Poisson approximates a binomial better when n (in the binomial) is large. Compare the Binomial(n,p) plot against to the Poisson(lambda = np) plot.