A review of Continuous Time MC

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Continuous Time Markov chains

Definition A *continuous time stochastic process* is a collection $\{Z_\alpha\}$ of random variables from a common sample space to a state space, where $\alpha \in \mathbf{R}^n$.

Definition A stochastic process $\{Y_t\}$ has the *Markov property* if for all measurable A, and $0 < s < t$

$$P(Y_t \in A | Y_r, 0 \leq r \leq s) = P(Y_t \in A | Y_s).$$

Definition A *continuous time Markov chain* is a continuous time stochastic process with the Markov property that changes value at a countable number of times, $0 = T_0 < T_1 < T_2 < \ldots$ with probability 1.

Definition For finite state space continuous time Markov chains, the *infinitesimal generator* of the chain is an $|S| \times |S|$ matrix with $Q(i,j) = q(i,j)$ for $j \neq i$, and $Q(i,i) = -\sum_{j \neq i} q(i,j)$.

Fact For $\{X_t\}$ a continuous time Markov chain with jumps at $0 = T_0 < T_1 < \ldots$,
1) $T_i - T_{i-1}$ are independent random variables, and $\{T_i - T_{i-1} | X_{T_{i-1}}\}$ is an exponential random variable with rate that only depends on $X_{T_{i-1}}$, and
2) $Y_i = X_{T_i}$ is a discrete time Markov chain called the *underlying discrete time chain* or *embedded chain* or *jump chain*.

Definition *Recurrence, transience, positive recurrence, and null recurrence* are defined the same way as for discrete time Markov chains. A continuous time Markov chain is *irreducible* if for all $x, y \in S$ and for some $t > 0$:

$$P(X_t = y | X_0 = x) > 0.$$

Fact A continuous time Markov chain is irreducible, recurrent, or transient if and only if the underlying discrete chain is (respectively) irreducible, recurrent, or transient. However, one can be positive recurrent while the other is null recurrent.

Ergodic Thm for continuous time

Definition Define the return time to a state x in a continuous time Markov Chain as follows:

$$R_x = \inf\{t > T_1 : X_t = x | X_0 = x\}.$$

Ergodic Thm for continuous time For an irreducible positive recurrent continuous time Markov chain on a countable state space, there exists a unique stationary distribution π with $\pi(i) > 0$. Also, π is the limiting distribution, and if R is the time for return to x starting from x, $E(R) = -1/[Q(x,x)\pi(x)]$. If the chain is null recurrent or transient, there is no stationary distribution π.

Poisson process

Definition A *Poisson process* is a process $\{X_t\}$ satisfying:

- $X_0 = 0$.
- The number of events during one time interval does not affect the number of events during a different time interval.
- The average rate at which events occur remians constant.
- Events occur once at a time.

Birth death chain

Definition A *birth death chain* is a continuous time Markov chain on state space $\{0, 1, 2, \ldots\}$ where $q(i, i+1) = \lambda_i$ for $i \geq 0$, $q(i, i-1) = \mu_i$ for all $i \geq 1$ and no other edges exist.

Fact A *Poisson process* is a birth death chain where $\lambda_i = \lambda$ and $\mu_i = 0$ for all i.

Definition The *Yule process* is the process cross between Poisson process and Branching process. Each individual present at time t splits into 2 during the time interval $(t, t + \Delta t)$ with probability $\lambda \Delta t + o(\Delta t)$ and $\mu = 0$.

Birth and death process

Theorem A birth death chain is transient if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} < \infty.$$

Birth and death process

Theorem

A birth death chain is positive recurrent if and only if

$$q = \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

(the term $n = 0$, the sum is equal to 1). Also the stationary distribution π is

$$\pi_i = \frac{\lambda_0 \lambda_2 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \cdot q^{-1}.$$

Chapman-Kolmogorov forward equations

Theorem $q(x,y)$ is the rate at which a continuous time Markov chain moves from x to y and if $q(x) = \sum_{y \neq x} q(x,y)$, then

$$\frac{d}{dt} p_t(x,y) = -q(y)p_t(x,y) + \sum_{z \neq y} q(z,y)p_t(x,z),$$

where $p_t(x,y) = P(X_t = y | X_0 = x) = P(X_{t+s} = y | X_s = x)$.

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Balance equations

Definition If $q(x,y)$ is the rate at which a continuous time Markov chain moves from x to y, then the *balance equations* are for all $x \in S$:

$$\pi(x) \sum_{y \neq x} q(x,y) = \sum_{y \neq x} \pi(y) q(y,x).$$

The *detailed balance equations* (aka *reversibility*) are for all $x, y \in S$:

$$\pi(x) q(x,y) = \pi(y) q(y,x).$$

$M/M/1$ **queue**

This is a birth and death process with $\lambda_n = \lambda$ and $\mu_n = \mu$.

$\sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^n < \infty$ iff it is transient. Thus $\mu < \lambda$ iff it is transient.

$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n < \infty$ iff it is positive recurrent. Thus $\lambda < \mu$ iff it is positive recurrent.

If $\lambda < \mu$ then the stationary distribution $\pi(n) = (1-\rho)\rho^n$ where $\rho = \lambda/\mu$.

The expected value of the length of the queue is $\lambda/(\mu - \lambda)$ if $\lambda < \mu$.

$M/M/k$ queue

This is a birth and death process with $\lambda_n = \lambda$ and

$$\mu_n = \begin{cases} k\mu & \text{if } n \geq k \\ n\mu & \text{if } n < k. \end{cases}$$

$\sum_{n=1}^{\infty} C \left(\frac{k\mu}{\lambda}\right)^n < \infty$ iff it is transient. Thus it is transient iff $k\mu < \lambda$.

$\sum_{n=0}^{\infty} C \left(\frac{\lambda}{k\mu}\right)^n < \infty$ iff it is positive recurrent. Thus $\lambda < k\mu$ iff it is positive recurrent.

$M/M/\infty$ queue

$M/M/\infty$ is never transient and always positive recurrent.

The stationary distribution is

$$\pi(n) = \frac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!}.$$