HOMEWORK 4

STA 624.01, Applied Stochastic Processes
Spring Semester, 2009

Due: Tuesday March 10th, 2009

Readings: All Chapter 2 of text

Regular Problems

1 (Lawler 2.3) Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p(x, x+1) = 2/3;\ p(x, 0) = 1/3.$$

Show that the chain is positive recurrent and give the limiting probability π.

2 (Lawler 2.4) Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities

$$p(x, x+2) = p,\ p(x, x-1) = 1-p,\ x > 0.$$

$$p(0, 2) = p,\ p(0, 0) = 1-p.$$

For which values of p is this a transient chain?

3 (Lawler 2.8) Given a braching process with the following offspring distributions, determine the extinction probability a:

(a)

$$p_0 = 0.25,\ p_1 = 0.4,\ p_2 = 0.35.$$

(b)

$$p_0 = 0.5,\ p_1 = 0.1,\ p_3 = 0.4.$$

(c)

$$p_0 = 0.91,\ p_1 = 0.05,\ p_2 = 0.01,\ p_3 = 0.01,\ p_6 = 0.01,\ p_13 = 0.01.$$

4 (Lawler 2.9) Consider the branching process with

$$p_0 = 0.5,\ p_1 = 0.1,\ p_3 = 0.4.$$

Suppose $X_0 = 1$.

(a) what is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not extinct in the first generation ($X_1 > 0$)?

(b) what is the probability that the population is extinct in the third generation ($X_3 = 0$), given that it did not extinct in the second generation ($X_2 > 0$)?

5 Prove that if the chain is irreducible and contains at least one recurrent state, then all states are recurrent.

6 Let $\xi_1, \xi_2, \cdots$ be a sequence of iid random variables. Let N be a discrete random variable and independent of $\xi_1, \xi_2, \cdots$. Define a discrete random variable X as

$$X = \begin{cases} 0 & \text{if } N = 0 \\ \xi_1 + \xi_2 + \cdots + \xi_N & \text{if } N \neq 0. \end{cases}$$

Prove that

$$Var(X) = \mathbb{E}(N)\sigma^2 + \mu^2 Var(N)$$

where $\mu = \mathbb{E}(\xi_i)$ and $\sigma^2 = Var(\xi_i)$.

Computer Problems

Computer Problem: Recurrence properties of two-dimensional lattices In lecture we showed that for a two dimension grid lattice simple random walk is null recurrent. Now we explore the behavior of simple random walk on a triangular lattice and a hexagonal lattice through simulation. In the triangular lattice, we tile the plane using equilateral triangles, in the hexagonal lattice, we tile the plane using regular hexagons. Check out
`http://library.thinkquest.org/16661/simple.of.regular.polygons/`
`regular.1.html?tqskip1=1&tqtime=0205`
for pictures of what these tilings (a.k.a. tesselations) look like. The grid lattice we studied in class is just the tiling with squares. There we discovered that the chain was null recurrent, so starting at the origin, the probability of return was 1. For the triangular lattice, make a conjecture about whether the chain is recurrent or transient. Let T be the time needed to return to the origin. Use two approaches to test recurrence. First plot estimates of $P(T > n)$ for large enough values of n to see if this approaches 0 or not. Second, estimate

$$P(X_n = \text{origin}|X_0 = \text{origin})$$

for a variety of values of n, and conjecture a relationship between these values and n. Use this conjecture to show that the expected number of visits to the origin is either finite or infinite. Note: In representing the current position in hexagonal and triangular lattices, it is easiest to just use two coordinates as in the square grid case. Use the first coordinate to represent number of moves "up and to the right" and the second coordinate to represent moves "to the right". The points of the hexagonal lattice are a subset of the points in the triangular lattice, so this framework can be used for both situations.

Branching process (Lawler 2.12) Consider the branching process with

$$p_0 = 1/3,\ p_1 = 1/3,\ p_2 = 1/3.$$

With the aid of computer, find the probability that the population dies out after n steps where $n = 20, 100, 200, 1000, 1500, 2000, 5000$. Do the same with the values

$$p_0 = 0.35,\ p_1 = 0.33,\ p_2 = 0.32.$$

Then do the same with with the values

$$p_0 = 0.32,\ p_1 = 0.33,\ p_2 = 0.35.$$