HOMEWORK 5

STA 624.01, Applied Stochastic Processes
Spring Semester, 2009

Due: Monday, March 24th, 2009

Readings: Section 3.3 of text

Regular Problems

1 (Lawler 3.3) Suppose X_t and Y_t are two independent Poisson processes with rate parameters λ_1 and λ_2, respectively, measuring of calls arriving at two different phones. Let $Z_t = X_t + Y_t$.

(a) Show that Z_t is a poisson process. What is the rate parameter for Z?

(b) What is the probability that the first call comes on the first phone?

(c) Let T be the first time that at least one call has come from each of the two phones. Find the density and distribution function of the random variable T.

2 Measurements in a telephone exchange show that, during business hours, a telephone call in progress ends with constant rate $\lambda = 0.5$ per minute.

(a) What is the distribution of the duration of a call?

(b) What is the probability that a call lasts no longer than one minute?

(c) What is the probability that a call that has lasted for one minute already, will be finished during the next minute?

3 (a) Consider N independent Poison processes, where the rate of the ith Poison process is λ_i for $1 \leq i \leq N$. What is the probability that the first arrival from any process belongs to the jth process?

(b) Bus riders arrive at a bus stop as a Poisson process with its rate λ. They wait for the $i \in \{\text{blue, red, green}\}$ line with probability p_i. Line i buses arrive as a Poisson process with its rate λ_i and pick up all passengers waiting for bus i. Construct a CTMC model of this scenario. Also list all possible infinitesimal rates. **Hint**: There are three processes here and the state space is $\mathbb{Z}_+^3$ where $Z_+ := \{0, 1, 2, \cdots\}$.

4 Suppose $T_1, \cdots, T_n$ are independent randam variables each exponential with rates $b_1, \cdots, b_n$, respectively. Let $T = \min\{T_1, \cdots, T_n\}$. Show (in details) that

$$P(T_1 = T) = \frac{b_1}{b_1 + \cdots + b_n}.$$

5 A substitution model describes the process from which a sequence of characters of a fixed size from some alphabet changes into another set of traits. The Jukes-Cantor model is one of them. Go to `http://en.wikipedia.org/wiki/Models_of_DNA_evolution` and

(1) Describe the model, i.e., what is the state space? Is it a CTMC? If so what is the rate matrix? In Biology, what does the model mean?

(2) What is the limiting distribution? Please write in etails.

(3) What is a distance between two sequences? Derive the distance under this model.

Computer Problem This computer problem has three parts. In the first part we will look at the Poisson process, in the second we will look at the Poisson distribution, and in the third we will examine the relationship between the Poisson distribution and the binomial distribution.

(a) The Poisson distribution is often used to model the number of events that occur in a fixed time interval or fixed spatial region when they are occurring "at random", like shooting stars.

Suppose you spend five minutes watching shooting stars, and you know that on average they fall at 3 per minute. Thus, the rate parameter for our five-minute time period is lambda=15. We expect, on average, to see 15 shooting stars per five minutes. Show what a Poisson process might look like by using a simulation by breaking that five-minute-long time interval into very small intervals of 1 second and plot them.

(b) Simulate 10,000 star-gazing five-minute periods all at once. Then estimate the count of how many shooting stars you would see in a five-minute stretch. ALso estimate its mean, standard deviation, and variance.

(c) In class we learned that Binomial(n, p) can be approximated by Poisson(λ) with $\lambda = np$, when n is very large. Let us check the approximation. Try different values of $n = 75$ and $n = 100$ with the same λ in (a). See how the Poisson approximates a binomial better when n (in the binomial) is large. Compare the Binomial(n,p) plot against to the Poisson(lambda = np) plot.