HOMEWORK 6

STA 624.01, Applied Stochastic Processes
Spring Semester, 2009

Due: Friday, April 14th, 2008

Readings: Chapter 3 of text

Regular Problems

1 (Lawler 3.7) Let X_t be a Markov chain with state space $\{1, 2, 3\}$ and rates $q_{1,2} = 1$, $q_{2,1} = 4$, $q_{2,3} = 1$, $q_{3,2} = 4$, $q_{1,3} = 0$, $q_{3,1} = 0$. Let

$$P_t(i, j) = P(Y_t = j|Y_0 = i).$$

Find the matrix P_t.

2 (Lawler 3.10) Consider the continuous-time Markov chain with the state space $\{1, 2, 3, 4\}$ and infinitesmal generator

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find the stationary distribution $\bar{\pi}$.

(b) Suppose that the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

(c) Again suppose that the chain starts in state 1. What is the expected amount of time until it changes to state 4?

3 (Lawler 3.11) Let X_t be a countinuous-time birth-dan-death process with the birth rate $\lambda_n = 1+(1/(n+1))$ and death rate $\nu_n = 1$. Is it process positive recurrent, null reccurent, or transient?

4 Consider Yule process with the rate λ (i.e., it is a BD process with birth rate $x\lambda$ and death rate $\mu = 0$ for any state $x \in S$). Let $Y_x = \inf\{t : X_t = x\}$ for $x \geq 2$. Compute $E(Y_x)$.

5 For a Markov chain, the global balance equations $\pi_j = \sum_{i \in S} \pi_i P(i, j)$ for all j characterise the stationary distribution(s). The equations

$$\pi_i P(i, j) = \pi_j P(j, i) \text{ for all } i \text{ and } j$$

are called the local balance equations.

(a) Prove that if the local balance equations are satisfied, so are the global ones. Hence, if π is a distribution that satisfies the local balance equations, it is a stationary distribution. Hint: Note that, for all j, $\sum_{i \in S} P(j, i) = 1$.

(b) Find out the right way of formulating local balance equations in continuous time, that is, for Markov processes. Show that, again, local balance implies global balance.

Our conclusion is that local balance equations may be used to find stationary solutions–and in fact they typically yield simpler equations than what global balance does–but it is a method that does not always work.

6 Let X_t is distributed Yule process with the birth raye λ. Compute $\mathbf{E}(X_t)$ and $\mathbf{V}(X_t)$.

Computer Problem: Branching Processes Consider a Lotka-Volterra predator-prey model where

$$X_t = \text{number of prey}$$
$$Y_t = \text{number of predators}$$

with moves:

move	rate
$(1, 0)$	aX_t
$(-1, 0)$	bX_tY_t
$(0, 1)$	cX_tY_t
$(0, -1)$	dY_t

Using the following parameters: $a = 1$, $b = .02$, $c = .01$, $d = 1$, $X_0 = 120$, $Y_0 = 40$, run the model forward five times. For each run plot two things. First, plot X_t and Y_t versus time up to about $t = 20$. Second, plot X_t versus Y_t over the same time frame.

Do the same two plots for a single run of length $t = 200$. For all your plots, you may abort your simulation if the predators go extinct. Just plot up to the time of extinction if that happens.