Name____

STA 624 Practice Exam 2

Applied Stochastic Processes

Spring, 2008

There are five questions on this test. DO use calculators if you need them. “And then a miracle occurs” is not a valid answer. There will be no bathroom break allowed. Please keep all prayers silent.

You have 50 minutes to complete this test. Please ask me questions if a question needs clarification.

Each question is worth the same number of points.

Question 1: Definitions

(a) What does it mean for a discrete time Markov chain to be the embedded (underlying) chain for a continuous-time Markov chain $\{X_t\}$?

(b) What is the Markov property for a continuous time Markov chain?

(c) What is to be time homogeous for a continuous time Markov chain?

(d) What is a Poisson process?

(e) What does it mean for a state x to be recurrent?

(f) What is to be irreducible for a continuous time Markov chain?

(g) What is a birth and death process?

(h) What is a time-reversible continuous time Markov chain?

Question 2: True or False

Mark whether each of the following states is true (T) or false (F). State a reason for each question.

(a) $M/M/\infty$ queue is always recurrent.

(b) A Yule process is positive recurrent.

(c) A poisson process is positive recurrent.

(d) Let $\{X_t, t \geq 0\}$ be Poisson processes with parameter λ. Define $Z_1 = 2X_t$. Z_1 is a Poisson process.

(f) Let $\{X_t, t \geq 0\}$ be Poisson processes with parameter λ. Define $Z_2 = X_t + k$ where $k > 0$. Z_2 is a Poisson process.

(g) Let $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ be independent Poisson processes with parameter λ and β. Define $\{Z_t, t \geq 0\}$ such that $Z_t = X_t + Y_t$. $\{Z_t, t \geq 0\}$ is a Poisson process.

(h) Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t, t \geq 0\}$ such that $Z_t = k \cdot X_t$ where $k > 1$ is a constant. $\{Z_t, t \geq 0\}$ is a Poisson process.

(i) Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t, t \geq 0\}$ such that $Z_t = X_t + k$ where $k > 0$ is a constant. $\{Z_t, t \geq 0\}$ is a Poisson process.

(j) Let $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ be independent Poisson processes with parameter λ and β. Define $\{Z_t, t \geq 0\}$ such that $Z_t = X_t - Y_t$. $\{Z_t, t \geq 0\}$ is a Poisson process.

(k) Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t, t \geq 0\}$ such that $Z_t = X_{k \cdot t}$, where $k > 0$ is a constant. $\{Z_t, t \geq 0\}$ is a Poisson process.

(l) Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter λ. Define $\{Z_t, t \geq 0\}$ such that $Z_t = X_{t^2}$. $\{Z_t, t \geq 0\}$ is a Poisson process.

Question 3: Theorems

(a) State the Ergodic Theorem for countable state space continuous time Markov chains.

(b) State Chapman-Komogorov equations.

(c) State Chapman-Komogorov forward equations.

(d) State Chapman-Komogorov backward equation.

(e) State the detailed balance equations.

(f) State the global balance equations.

Questions 4: Examples (Note: you do not have to prove that your example meets the required criteria, you just have to present it.)

(a) Give an example of a transient birth-death chain

(b) Give an example of a positive recurrent birth-death chain

(c) Give an example of a positive recurrent M/M/5 queue.

(d) Give an example of a M/M/1 queue such that the expected value of the length of queue is 1/2.

(e) Give an example of a continuous time Markov chain with the underlying discrete time Markov chain with $S = \{1, 2, 3, 4\}$ and the transition probability

$$D = \begin{pmatrix} 0 & 0.2 & 0.3 & 0.5 \\ 0.3 & 0 & 0.1 & 0.6 \\ 0 & 0.5 & 0 & 0.5 \\ 0.4 & 0.1 & 0.5 & 0 \end{pmatrix}$$

(f) Give an example of a continuous time Markov chain with the underlying discrete time Markov chain with $S = N \cup \{0\}$ and the transition probability $p(x, x+1) = 1/2$, $p(x, x-1) = 1/2$, for $x \geq 1$ and $p(0, 1) = 1$.

Question 5: Calculations

(1) Let B_i be iid random variables such that

$$B_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases},$$

where $0 < p < 1$. Let $N(t)$ be a Poisson process independent of B_i with constant rate λ. Define

$$X_t = \begin{cases} \sum_{i=1}^{N(t)} B_i & \text{if } \sum_{i=1}^{N(t)} B_i \geq 0 \\ 0 & \text{else} \end{cases}.$$

(a) Is $\{X_t,\, t \geq 0\}$ a continuous time Markov chain? If so describe it specifically.

(b) Under what conditions does it become positive recurrent?

(c) Under the condition (b), calculate the stationary distribution.

(2)

(a) Consider N independent Poison processes, where the rate of the ith Poison process is λ_i for $1 \leq i \leq N$. What is the probability that the first arrival from any process belongs to the jth process?

(b) Let $\{X_t,\, t \geq 0\}$ be independent Poisson processes with its rate λ and let $\{T_k : k \geq 0\}$ be the kth arrival time. Compute

$$E[T_k].$$

(c) A birth dead process has the birth rate $\lambda_n = 1/\sqrt{n+1}$ and death rate $\mu_n = 1/\sqrt{n}$. Is this transient? Positive recurrent? Or null recurrent?

(2) Consider a Poisson process $\{X_t, t \geq 0\}$ with a parameter λ.
(a) Write down the rate matrix Q for the process.

(b) Let $P_{i,j}(t)$ be the transition probability from a state i to state j. Write down Kolmogorov's forward and backward equations for this process, $\{X_t, t \geq 0\}$.

(c) What is the probability that k events occur in a time interval $(s, s+t]$ where $s, t > 0$ for the Poisson process, $\{X_t, t \geq 0\}$.