HOMEWORK 1

STA 624.01, Applied Stochastic Processes
Spring Semester, 2010

Due: Tuesday, February 9, 2010

Readings: Chapter 1 of text, Tutorial 2 on MATLAB

Note: the computer problems require simulation and the use of a computer. You are allowed (encouraged, even) to use a computer in solving the other problems as well.

When giving numerical answers, please give results to four siginificant figures unless they are integer answers. So $1/2 = .5000$ for example. Also box your numerical answers.

Regular Problems

1 Suppose P is a $s \times s$ stochastic matrix (i.e., the row sums equal to one) and

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{pmatrix}.$$

Show that P^2 is a $s \times s$ stochastic matrix and P^n is also a $s \times s$ stochastic matrix for all positive integer n.

2 Three different Markov chains are defined by the following transition matrices:

1.
$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

2.
$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.
$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

(a) Draw a transition graph for each chain. Is the MC irreducible?

(b) Identify the communication classes and classify them as periodic or aperiodic (if it is a periodic state the periodicity of it), transient or recurrent.

3 Assume this chain has transition probability matrix

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

(a) Show that the chain is irreducible, recurrent, and periodic. What is the periodicity of the chain?

(b) Find the unique stationary probability distribution.

4 A sequence of electrical impulses passes a measurement instrument that stores the largest value measured so far. Assume that the impulses at time points $0, 1, 2, 3, \cdots$ can be modelled as independent random variables $Y_0, Y_1, Y_2, Y_3, \cdots$ with a uniform distribution on $\{1, 2, 3, 4, 5\}$. Thus, if $X_1, X_2, X_3, \cdots$ are the values stored at time points $0, 1, 2, 3, \cdots$, then

$$X_n = \max(Y_0, Y_1, Y_2, \cdots, Y_n) \text{ for } n = 0, 1, 2, 3, \cdots .$$

Motivate that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain and write down the transition probability matrix.

5 Suppose that two unbiased coins are tossed repeatedly and after each toss the accumulated number of heads and tails that have appeared on each coin is recorded. Let X_n be the difference in the accumulated number of heads on coin A and coin B after the nth toss, i.e., $X_n =$ (Total number of heads on coin A) $-$ (Total number of heads on coin B). Thus, the state space $S = \{0, \pm 1, \pm 2, \cdots\}$. Show that the zero state, where the total number of heads equal on each coin, is null recurrent.

Computer Problems

For this problem, please print out all code used and all results.

Consider the Markov chain I demonstrated in classes. We have two strings. We have two moves; one is twist and one is rotate. These moves correspond to the following maps:

$$\begin{aligned} x &\to x+1 \qquad \text{for twist} \\ x &\to -\frac{1}{x} \qquad \text{for rotate.} \end{aligned}$$

When two strings are untangled and aligned nicely we have $x = 0$ (this is always true!!!).

Let X_n be the number x after n moves and we start $X_0 = 0$.

The state space is the set of all rational numbers $\mathbb{Q}$. Each move is defined as follows:

$$\begin{aligned} P(X_{n+1} = x+1 | X_t = x) &= p, \quad \text{if } x \neq 0 \\ P(X_{n+1} = -1/x | X_n = x) &= 1-p, \quad \text{if } x \neq 0 \\ P(X_{n+1} = 1 | X_n = 0) &= 1 \end{aligned}$$

a) Write code for simulating this Markov chain.

b) Find the limiting distribution when $n = 10$ and $n = 15$ for $p = 0.2$, 0.5, 0.6 by simulating the chain multiple times starting from $X_0 = 0$.

c) For $n = 5$ with $p = 0.2$, 0.5, 0.6, estimate the expected number of steps needed to return to 0 starting at 0.